Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\, x + \ a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \ 2\, a_2\, x + 3\, a_3\, x^2 + 4\, a_4\, x^3 + \cdots$$

$$f''(x) = 2\, a_2 + \ (3\times2)a_3\, x + (4\times3)a_4\, x^2 + \cdots$$

$$f'''(x) = (3\times2)a_3 \ + (4\times3\times2)a_4\, x + \cdots$$

$$f'^{v}(x) = (4\times3\times2)a_4 + \cdots$$

$f(0) = a_o$

$f'(0) = a_1$

$f''(0) = 2\, a_2$

$f'''(0) = (3\times2)\, a_3$

$f'^{v}(0) = (4\times3\times2)\, a_4$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(0)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i$$

Example:

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4; \; f^{(n)}(1) = 0 \; \forall \, n \; \geq 3$$

Evaluate $f(4)$

Taylor Series

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $h = x - x_o$

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

error = exact - approximation

Taylor series with remainder

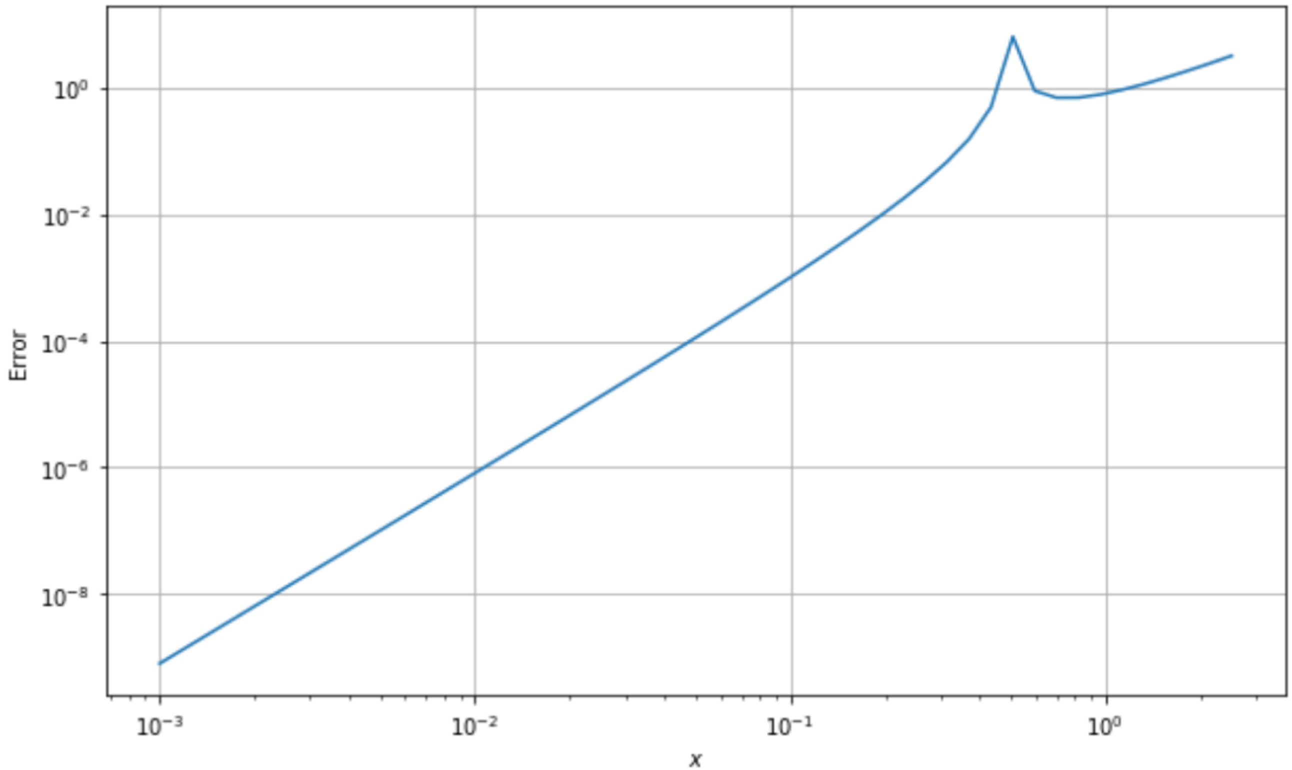
Graphical representation:

Example:

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$



10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
Error
10^{-3}
10^{-2}
10^{-1}
10^0
x

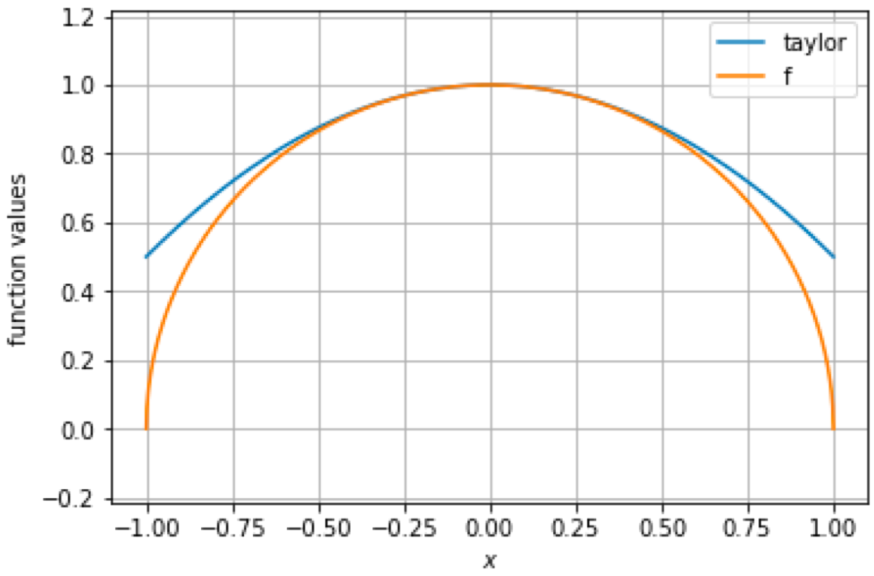
Example:

Given the function

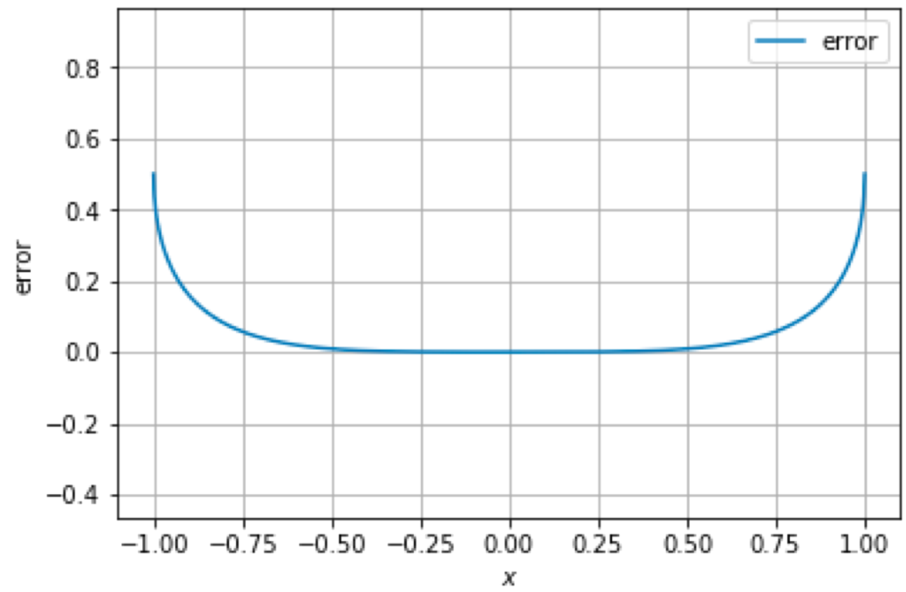
$$f(x) = \sqrt{-x^2 + 1}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

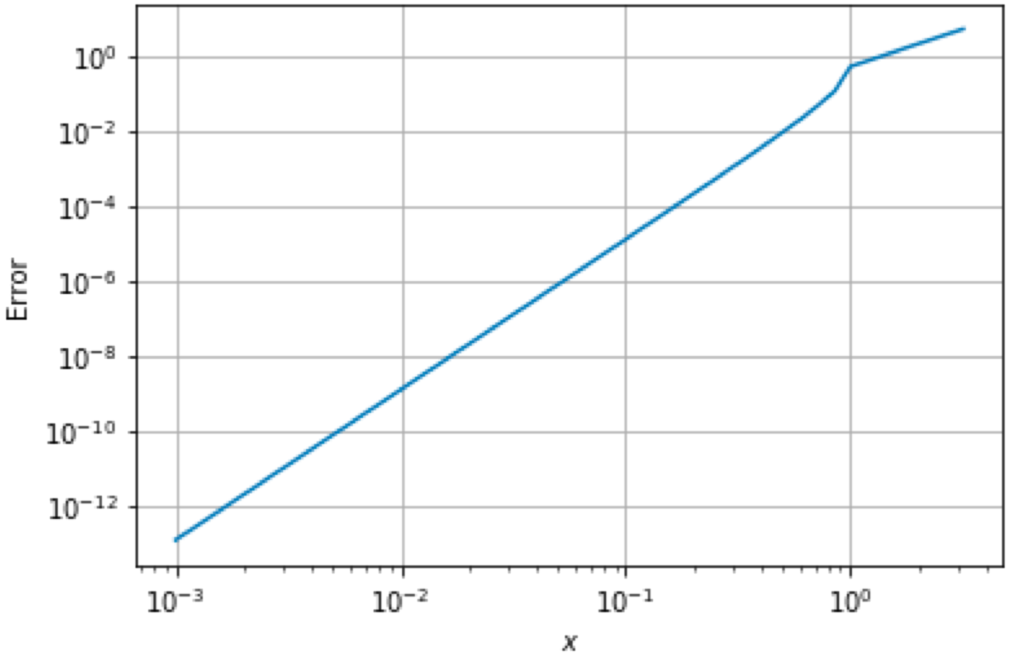
$$f(x) = \sqrt{-x^2 + 1}$$



taylor
f
function values
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x



error
error
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x



Error
x
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-3}
10^{-2}
10^{-1}
10^{0}

Example:

Error Order for Taylor series

1 point

The series expansion for e^x about 2 is

$$\exp(2) \cdot \left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \dots\right).$$

If we evaluate e^x using only the first four terms of this expansion (i.e. only terms up to and including $\frac{(x-2)^3}{3!}$), then what is the error in big-O notation?

Choice*

A) $O(x^4)$
B) $O(x^5)$
C) $O(x^3)$
D) $O((x-2)^3)$
E) $O((x-2)^4)$

Demo "Taylor of exp(x) about 2"

Making error predictions

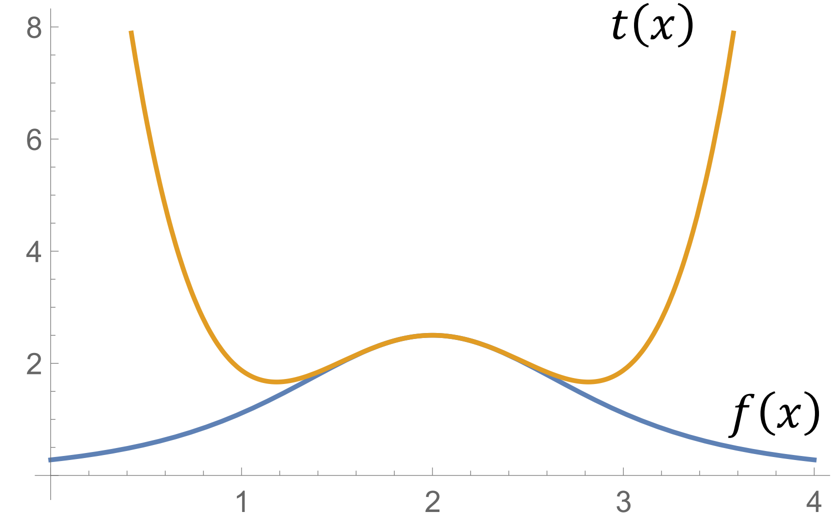
Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

What is the Taylor truncation error for $h_2 = 0.25$?

Using Taylor approximations to obtain derivatives

Let's say a function has the following Taylor series expansion about $x = 2$.

$$f(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4 - \frac{5}{4}(x-2)^6 + \frac{25}{32}(x-2)^8 + O((x-2)^9)$$



Iclicker question

A function $f(x)$ is approximated by the following Taylor polynomial of degree $n = 2$ about $x = 2\pi$

$$t_2(x) = 39.4784 + 12.5664\,(x - 2\pi) - 18.73922\,(x - 2\pi)^2$$

Determine an approximation for $f'(6.1)$

A) 18.7741
B) 12.6856
C) 19.4319
D) 15.6840

Finite difference approximation

For a given smooth function $f(x)$, we want to calculate the derivative $f'(x)$ at $x = 1$.

Suppose we don't know how to compute the analytical expression for $f'(x)$, but we have available a code that evaluates the function value:

```
def f(x):
    # do stuff here
    feval = ...
    return feval
```

Can we find an approximation for the derivative with the available information?

Demo: Finite Difference

$f(x) = e^x - 2$

We want to obtain an approximation for $f'(1)$

$dfexact = e^x$

$$dfapprox = \frac{e^{x+h} - 2 - (e^x - 2)}{h}$$

$error(h) = abs(dfexact - dfapprox)$

$$error < \left| f''(\xi)\frac{h}{2} \right|$$

truncation error

Demo: Finite Difference

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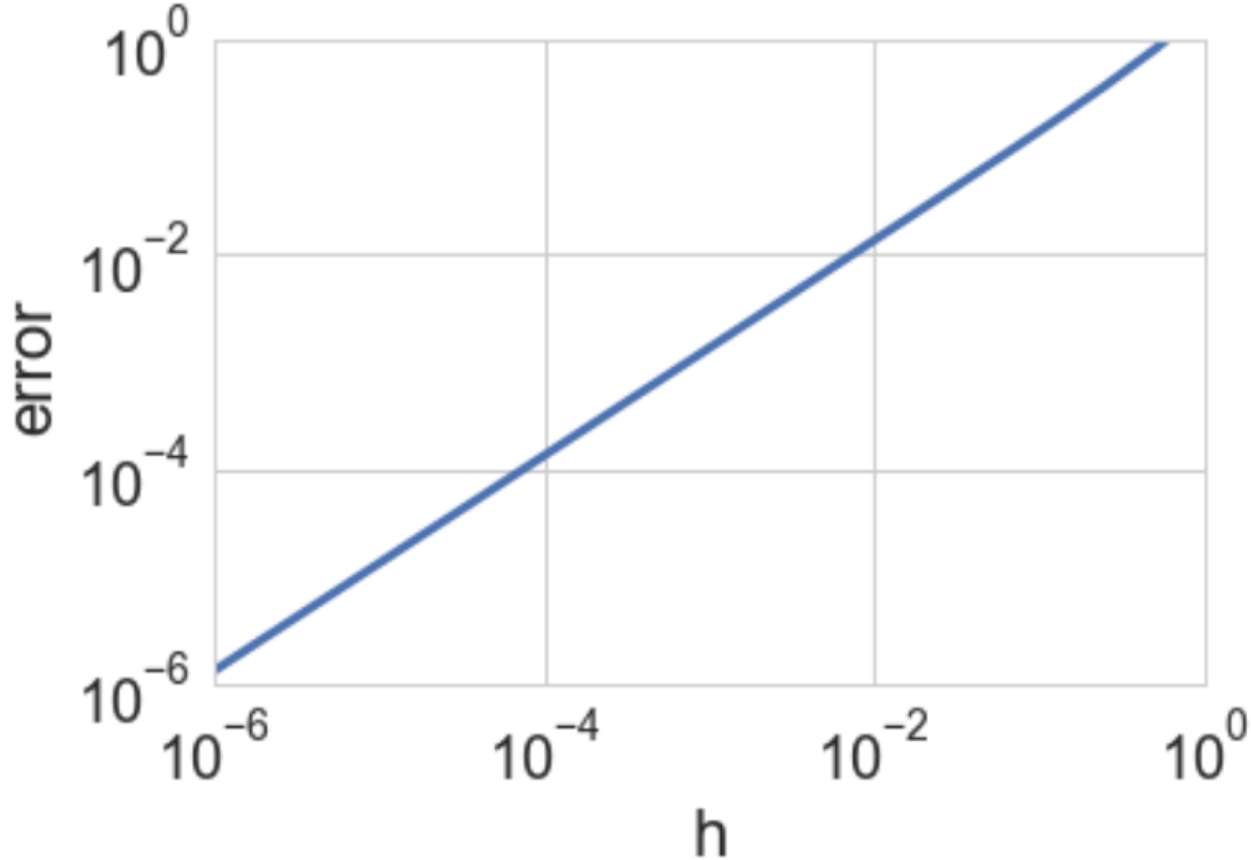
truncation error

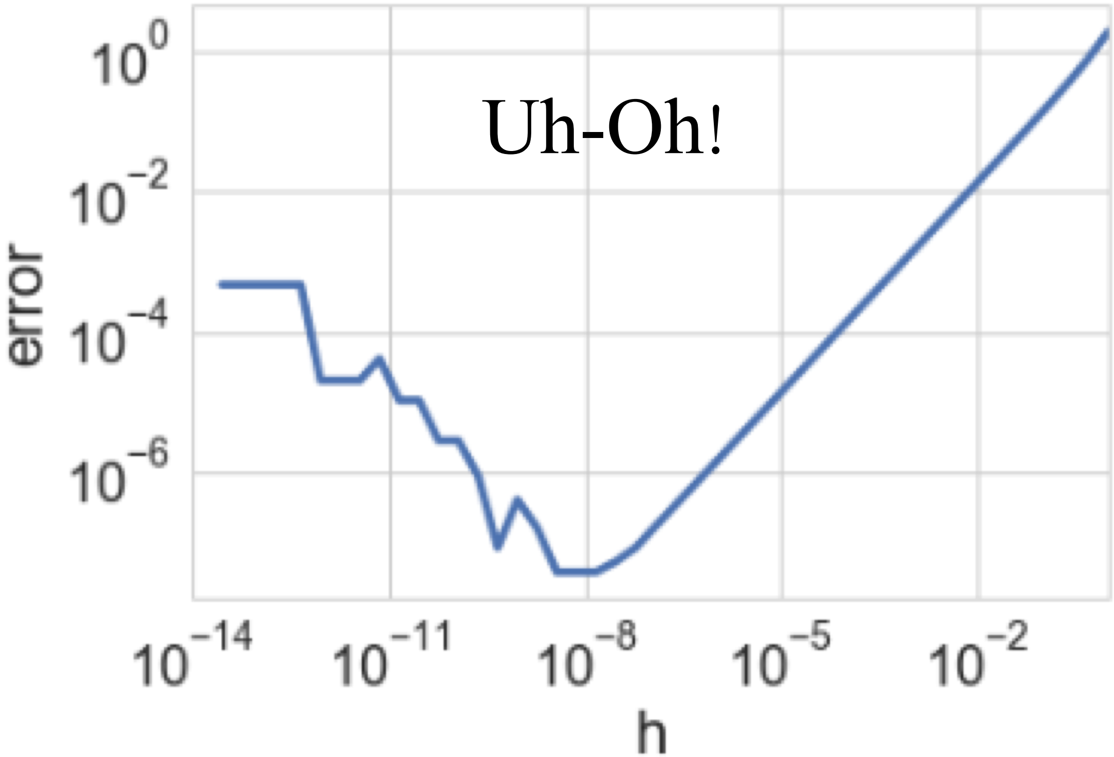
h	$error$
1.000000E+00	1.952492E+00
5.000000E-01	8.085327E-01
2.500000E-01	3.699627E-01
1.250000E-01	1.771983E-01
6.250000E-02	8.674402E-02
3.125000E-02	4.291906E-02
1.562500E-02	2.134762E-02
7.812500E-03	1.064599E-02
3.906250E-03	5.316064E-03
1.953125E-03	2.656301E-03
9.765625E-04	1.327718E-03
4.882812E-04	6.637511E-04
2.441406E-04	3.318485E-04
1.220703E-04	1.659175E-04
6.103516E-05	8.295707E-05
3.051758E-05	4.147811E-05
1.525879E-05	2.073897E-05
7.629395E-06	1.036945E-05
3.814697E-06	5.184779E-06
1.907349E-06	2.592443E-06

Demo: Finite Difference



$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Should we just keep decreasing the perturbation h, in order to approach the limit $h \to 0$ and obtain a better approximation for the derivative?

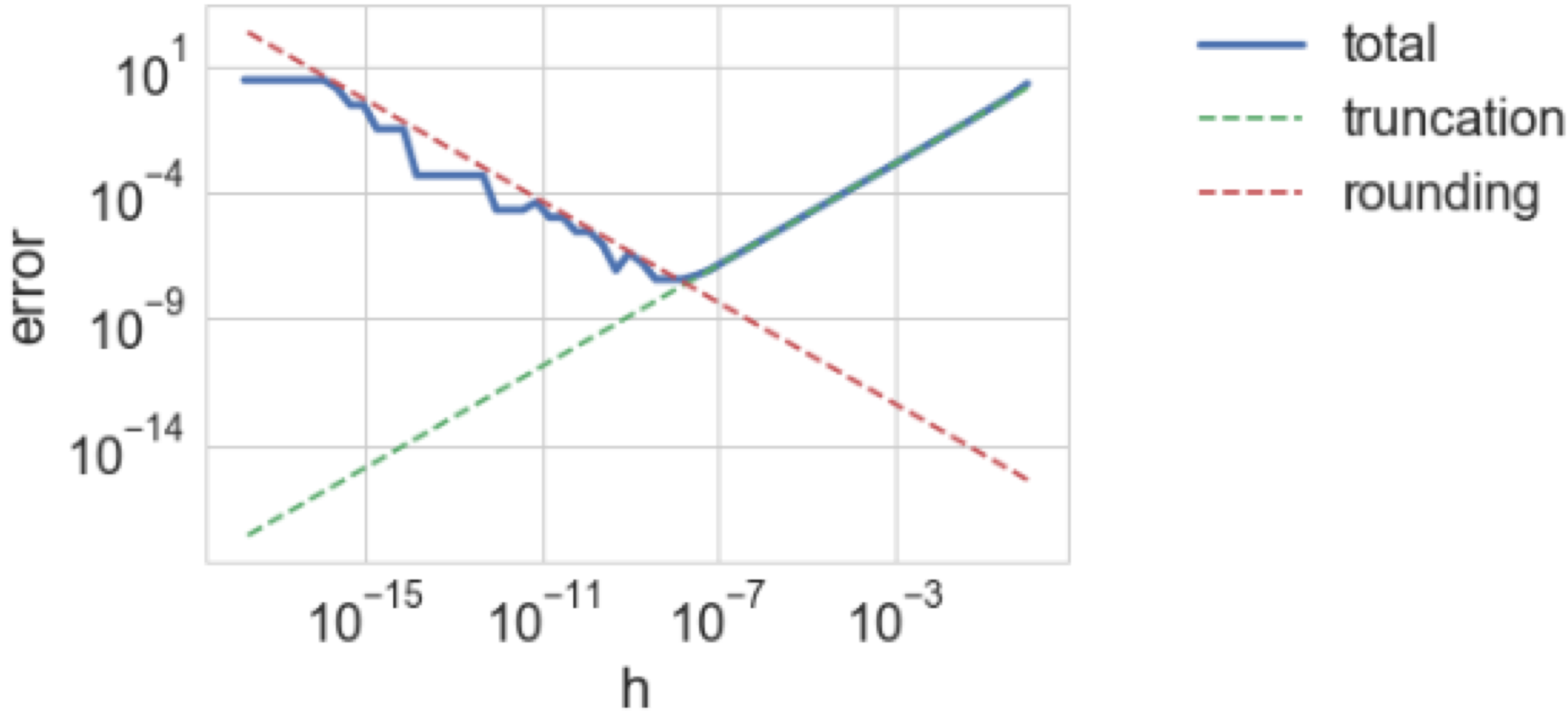


What happened here?

$$f(x) = e^x - 2$$

$$f'(x) = e^x \rightarrow f'(1) \approx 2.7$$

$$f'(1) = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$



Truncation error: $$error \sim M\frac{h}{2}$$

Rounding error: $$error \sim \frac{2\epsilon}{h}$$