Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \; 2\, a_2\, x + 3\, a_3\, x^2 + 4\, a_4\, x^3 + \cdots$$

$$f''(x) = 2\, a_2 + \; (3\times2) a_3\, x + (4\times3) a_4\, x^2 + \cdots$$

or $f^{(i)} = i!\, a_i$

$\boxed{a_i = f^{(i)}/i!}$

$$f'''(x) = (3\times2) a_3 \; + (4\times3\times2) a_4\, x + \cdots$$

$f^{(i)} = \left(i \times (i-1) \times (i-2) \times \ldots \times 1\right) a_i$

$$f'^{v}(x) = (4\times3\times2) a_4 + \cdots$$

$f(0) = a_o$

$f'(0) = a_1$

$f''(0) = 2\, a_2$

$f'''(0) = (3\times2)\, a_3$

$f'^{v}(0) = (4\times3\times2)\, a_4$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + \ a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

$$\Rightarrow \boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!} x_i}$$

→ approximate function values

→ approximate derivatives

→ estimating errors

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(0)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i$$

Example:

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4; \; \underline{\underline{f^{(n)}(1) = 0}} \; \forall \, n \; \geq 3$$

Evaluate $f(4)$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cancel{\frac{f'''(x_0)}{3!}(x - x_0)^3} + \cdots$$

Make $x = 4$ and $x_0 = 1$

$$f(4) = f(1) + f'(1)(4-1) + \frac{f''(1)}{2}(4-1)^2 = 2 + (-3)(4-1) + \frac{4}{2}(4-1)^2$$

$$= 2 - 9 + 18 \Rightarrow \boxed{f(4) = 11}$$

Taylor Series

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $\overbrace{h = x - x_o}^{x = h + x_0}$

$$f(x_0+h) = f(x_0) + f'(x_0)\,h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots$$

$$\underbrace{f(x_0+h)}_{\text{exact}} = \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}h^i}_{\text{truncated part (Taylor approximation of degree } n)} + \underbrace{\sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!}h^i}_{\text{what we are neglecting}}$$

$f(x)$ $t_n(x)$ error

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

error = exact - approximation

$$\text{error} = f(x) - t_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i$$

$$= \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!} h^{n+1}}_{\text{dominant term}} + \frac{f^{(n+2)}(x_0)}{(n+2)!} h^{n+2} + \cdots$$

dominant term when $h \to 0$ (or $x \to x_0$)

$$\text{error} \leq M h^{n+1} \quad \text{or} \quad \text{error} = O(h^{n+1})$$

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

error = exact - approximation

Remainder Theorem : $R_n(x) = f(x) - t_n(x)$

$$= \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i$$

$$\boxed{R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!} (\xi - x_0)^{n+1}}$$

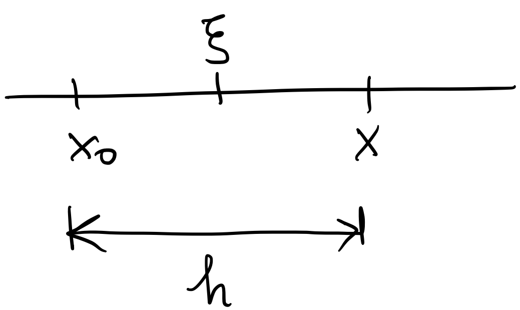
where $\xi \in (x_0, x)$

since $|\xi - x_0| \leq |h|$

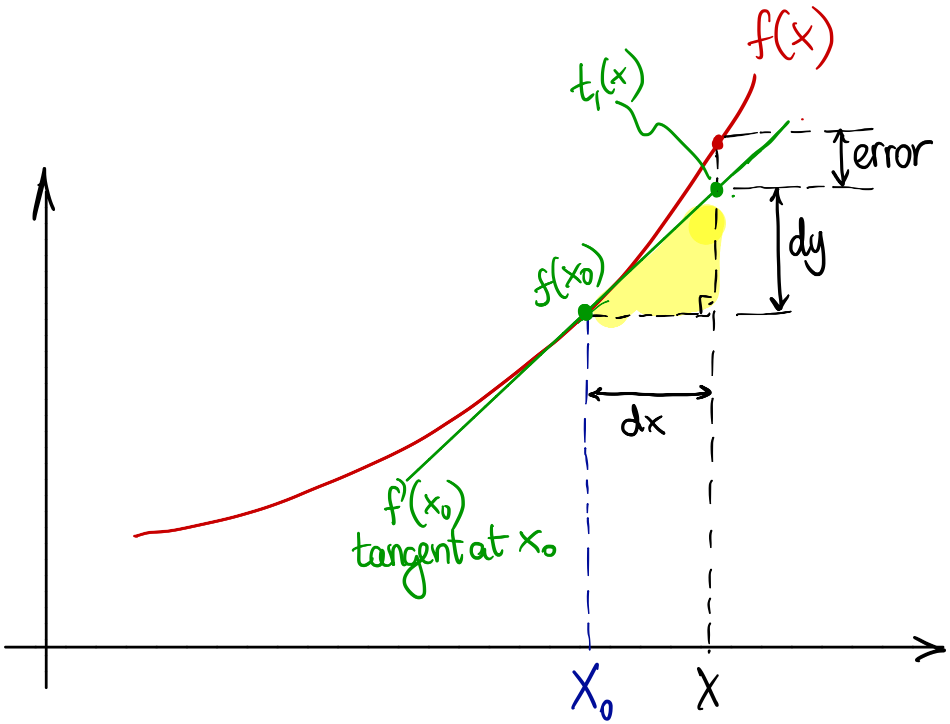
$$|R_n| \leq \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \right|$$

note

$$M = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|$$



Graphical representation:



$f'(x_0) = \frac{dy}{dx} \Rightarrow dy = f'(x_0)(x - x_0)$

$t_1(x) = f(x_0) + f'(x_0)(x - x_0)$ ✓ :)

$error = f(x) - t_1(x) = \text{Remainder}$

$error \leq \frac{f''(\xi)(x-x_0)^2}{2!} \qquad \xi \in (x_0, x)$

$\boxed{error = O(h^2)}$

suppose interval is reduced by half.
what happens to the error?

$e_1 — h_1^2$

$e_2 — \left(\frac{h_1}{2}\right)^2$

$\frac{e_1}{e_2} = 2^2$

$\boxed{e_2 = \frac{e_1}{4}}$

Example:

Demo

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

Given the function : $f(x) = \frac{1}{20x - 10}$

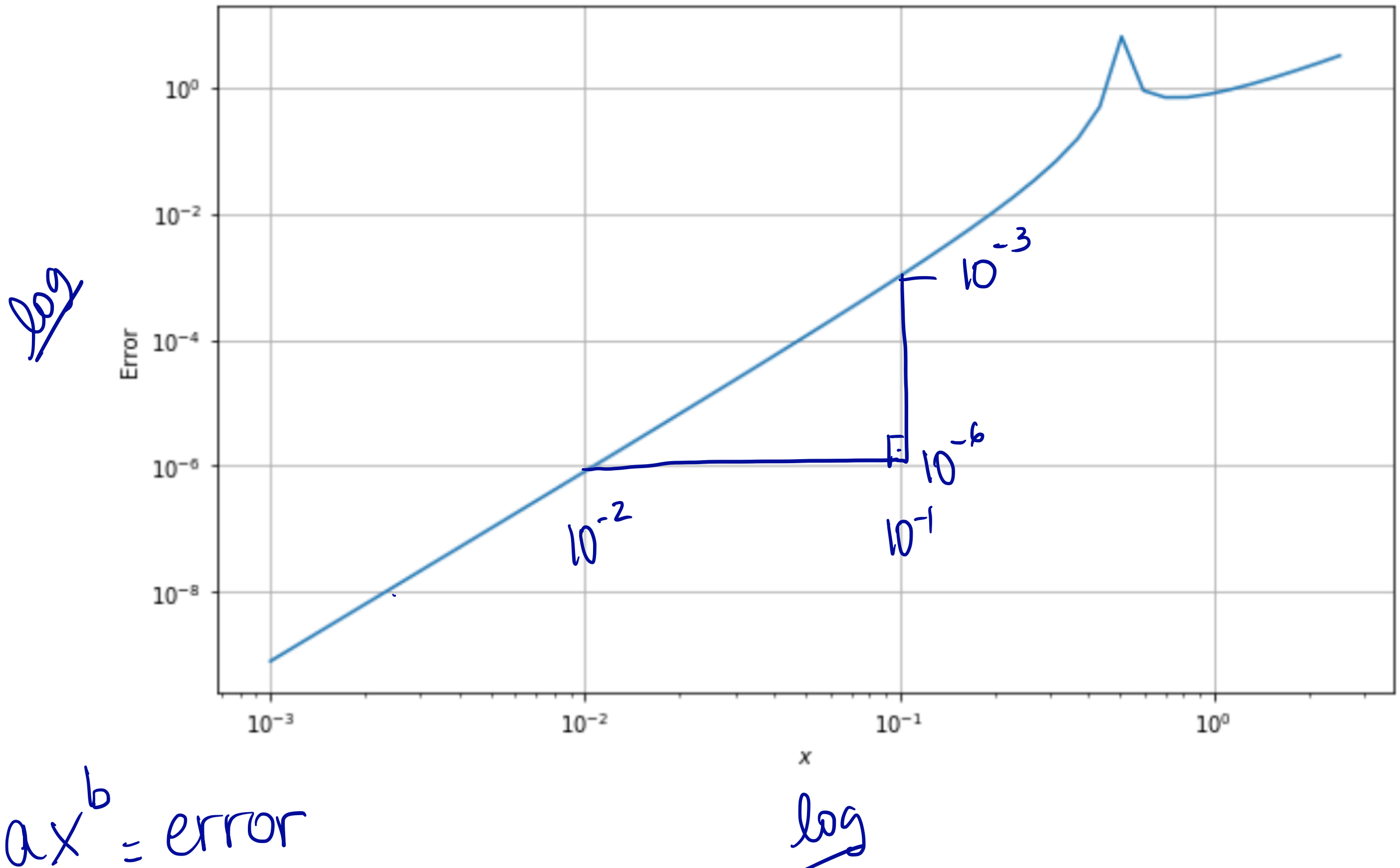
write the Taylor approximation of degree 2 about $x_0 = 0$

$f'(x) = \frac{-1(20)}{(20x-10)^2}$; $f'(0) = \frac{-20}{(-10)^2} = -\frac{1}{5}$

$f''(x) = \frac{+20(20x-10)2(20)}{(20x-10)^4} = \frac{-800}{(20x-10)^3}$ $\quad f''(0) = \frac{-800}{1000} = -4/5$

$$\boxed{t_2(x) = -\frac{1}{10} - \frac{1}{5}x - \frac{1}{2}\left(\frac{4}{5}\right)x^2}$$

$|R_2(x)| \leq \left|\frac{f'''(0)}{3!} x^3\right|$ $\quad$ error $= O(x^3)$



$$y = ax^b = \text{error}$$

$$\log(y) = \log(a) + \underbrace{b}_{\text{slope!}} \log(x)$$

$$b = \frac{\log 10^{-3} - \log 10^{-6}}{\log 10^{-1} - \log 10^{-2}} = \frac{-3+6}{-1+2} = \frac{3}{1} \Rightarrow \text{error} = O(x^3) \checkmark$$

Example:

DEMO

Given the function

$$f(x) = \sqrt{-x^2 + 1}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$f(0) = 1$$

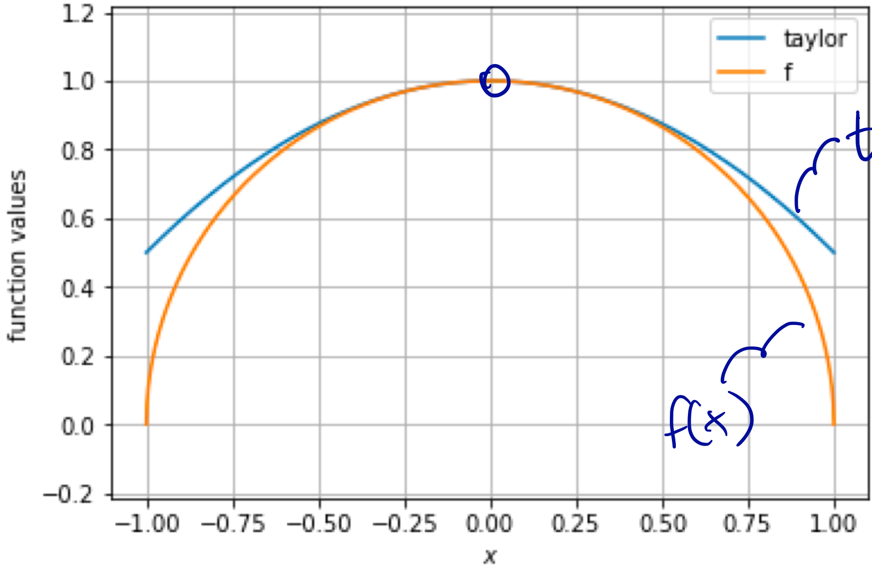
$$f'(x) = \frac{1}{\cancel{2}}(-x^2+1)^{-1/2}(-\cancel{2}x) = -x(1-x^2)^{-1/2} \longrightarrow f'(0) = 0$$

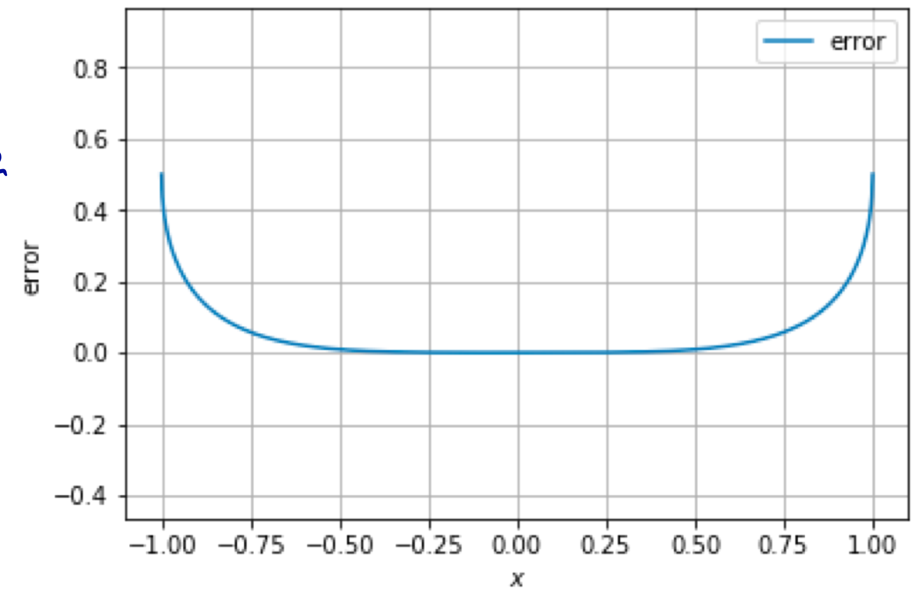
$$f''(x) = -\frac{1}{2}x(1-x^2)^{-3/2}(-2x) - (1-x^2)^{-1/2} \longrightarrow f''(0) = -1$$

$$\hat{f}(x) = 1 - \frac{1}{2}(x)^2 \quad \text{or} \quad \boxed{t_2(x) = 1 - \frac{x^2}{2}}$$

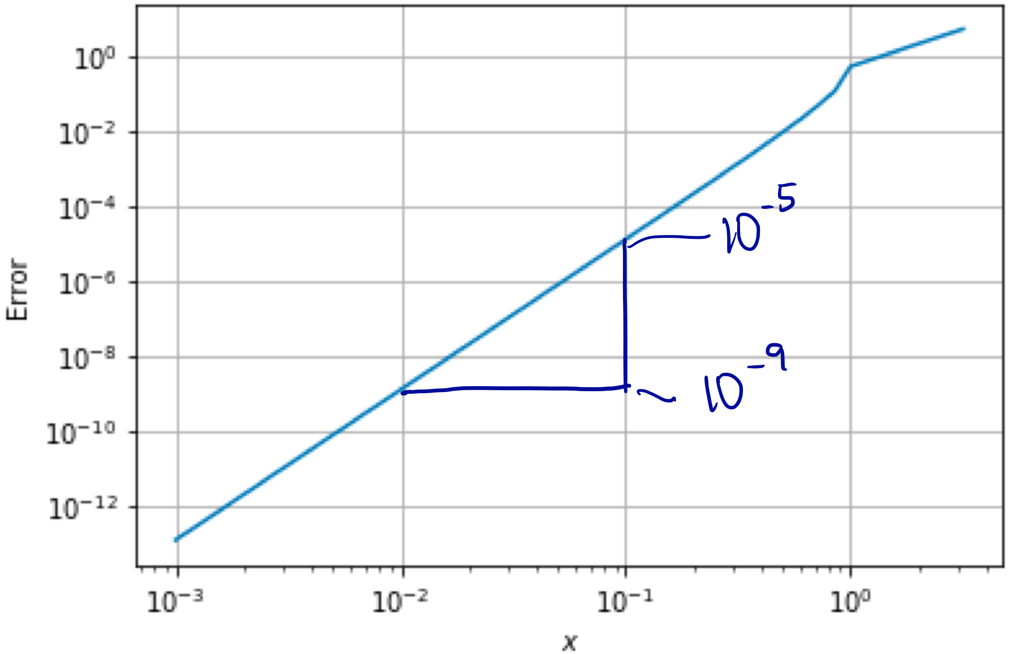
$$f(x) = \sqrt{-x^2 + 1}$$

$$\text{error} = t_2 - f(x)$$





- "good" approximation close to $x_0 = 0$
- error increases when x moves away from x_0
- use log-log plot to better visualize what is happening close to x_0.



$$\text{error} = t_2(x) - f(x)$$

$$f(x) = \sqrt{-x^2+1}$$

$$t_2(x) = 1 - x^2/2$$

$$|R_2| \leq \left| \frac{f'''(\xi)}{3!} h^3 \right| = O(h^3)$$

here $h = x - x_0 = x$

Let's get Big-O of error from the plot!

$$\text{slope} = \frac{\log(10^{-5}) - \log(10^{-9})}{\log(10^{-1}) - \log(10^{-2})} = \frac{-5+9}{-1+2} = 4 \implies \text{error} = O(h^4)$$

what happened here!

$\rightarrow f'''(x) = 0$ hence the next term that is not zero is $f^{IV}(x)$

Example:

DEMO

Error Order for Taylor series

1 point

The series expansion for e^x about 2 is

$$\underbrace{\exp(2)\cdot\left(1+(x-2)+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}}_{t_3(x)}+\ldots\right).$$

If we evaluate e^x using only the first four terms of this expansion (i.e. only terms up to and including $\frac{(x-2)^3}{3!}$), then what is the error in big-O notation?

$$\text{error} = e^2\left[\frac{(x-2)^4}{4!}+\frac{(x-2)^5}{5!}+\ldots\right]$$

$$\text{error} = O((x-2)^4)$$

Choice*

A) $O(x^4)$

B) $O(x^5)$

C) $O(x^3)$

D) $O((x-2)^3)$

E) $O((x-2)^4)$ → this is tightest bound.

all are valid options!

★ $e \le M(x-2)^4$

as $x \to 2$, e becomes smaller

★ $e \le M x^4$

as $x \to 2$, $e(x)$ does not show asymptotic behavior

Demo "Taylor of exp(x) about 2"