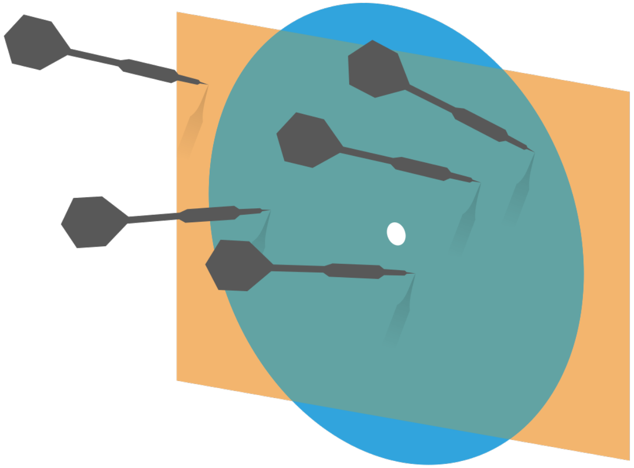
Probability and Statistics for Computer Science



Credit: wikipedia

Can we call e the exciting e ?

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.29.2020

Last time

Objectives

- Poisson distribution
- Continuous Random Variable
- Uniform Continuous distribution
- Exponential distribution

Motivation for a model called Poisson Distribution

- What's the probability of the **number of incoming customers (k)** in an hour?
- It's widely applicable in physics and engineering both for modeling of time and space.

Simeon D. Poisson Credit: wikipedia
(1781-1840)

Poisson Distribution

- A discrete random variable X is called **Poisson** with intensity **λ** (λ>0) if

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for integer $k \geq 0$

Simeon D. Poisson
(1781-1840)

λ *is the* **average rate** *of the event's occurrence*

Poisson Distribution

- **Poisson** distribution is a valid pdf for

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda}$$

Simeon D. Poisson
(1781-1840)

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

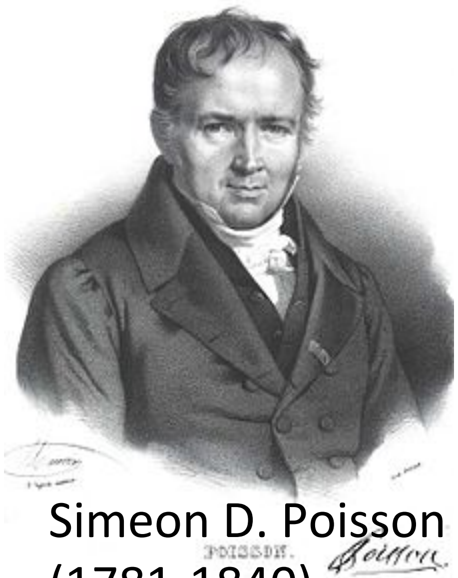
for integer $k \geq 0$

λ *is the* **average rate** *of the event's occurrence*

Poisson Distribution

- **Poisson** distribution is a valid pdf for

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = 1$$



Simeon D. Poisson
(1781-1840)

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for integer $k \geq 0$

λ *is the* **average rate** *of the event's occurrence*

Expectations of Poisson Distribution

- The expected value and the variance are wonderfully the same! That is **λ**

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for integer $k \geq 0$

Simeon D. Poisson
(1781-1840)

$$\boxed{\begin{aligned} E[X] &= \lambda \\ var[X] &= \lambda \end{aligned}}$$

Examples of Poisson Distribution

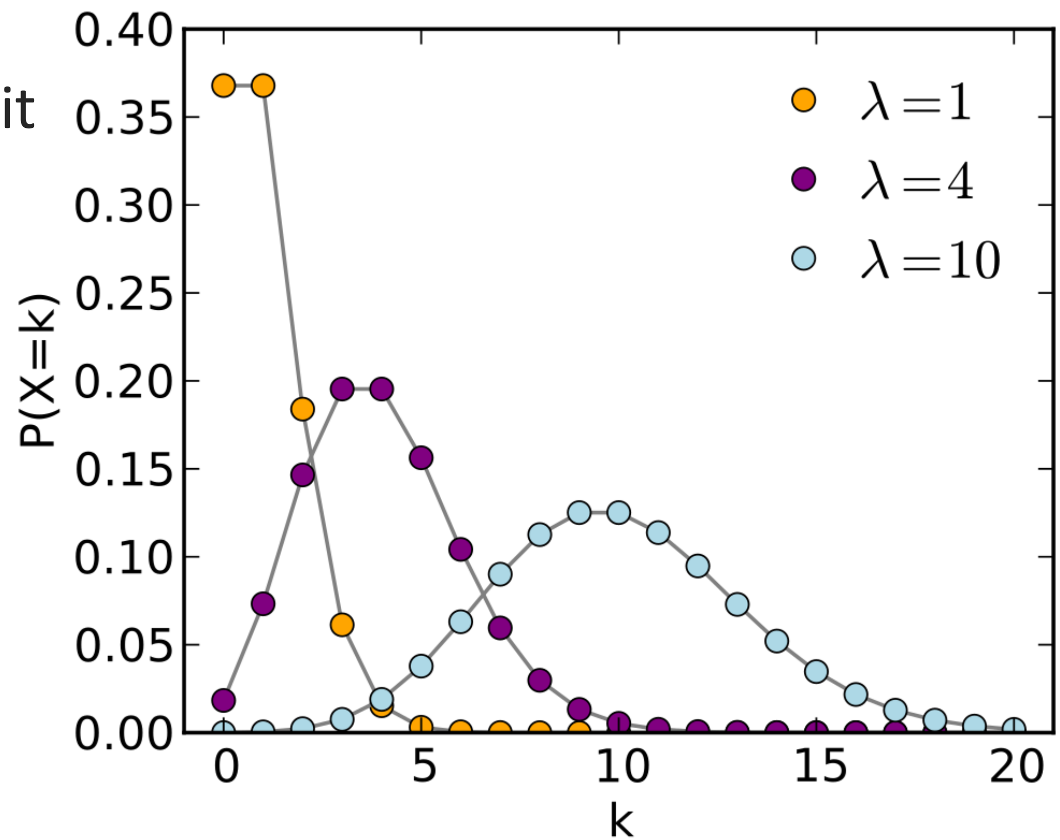
- How many calls does a call center get in an hour?
- How many mutations occur per 100k nucleotides in an DNA strand?
- How many **independent** incidents occur in an interval?

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for integer $k \geq 0$

Poisson Distribution: call center

- If a call center receives 10 calls per hour on average, what is the probability that it receives 15 calls in a given hour?
- What is **λ** here?
- What is P(k=15)?



Credit: wikipedia

Q. Poisson Distribution: call center

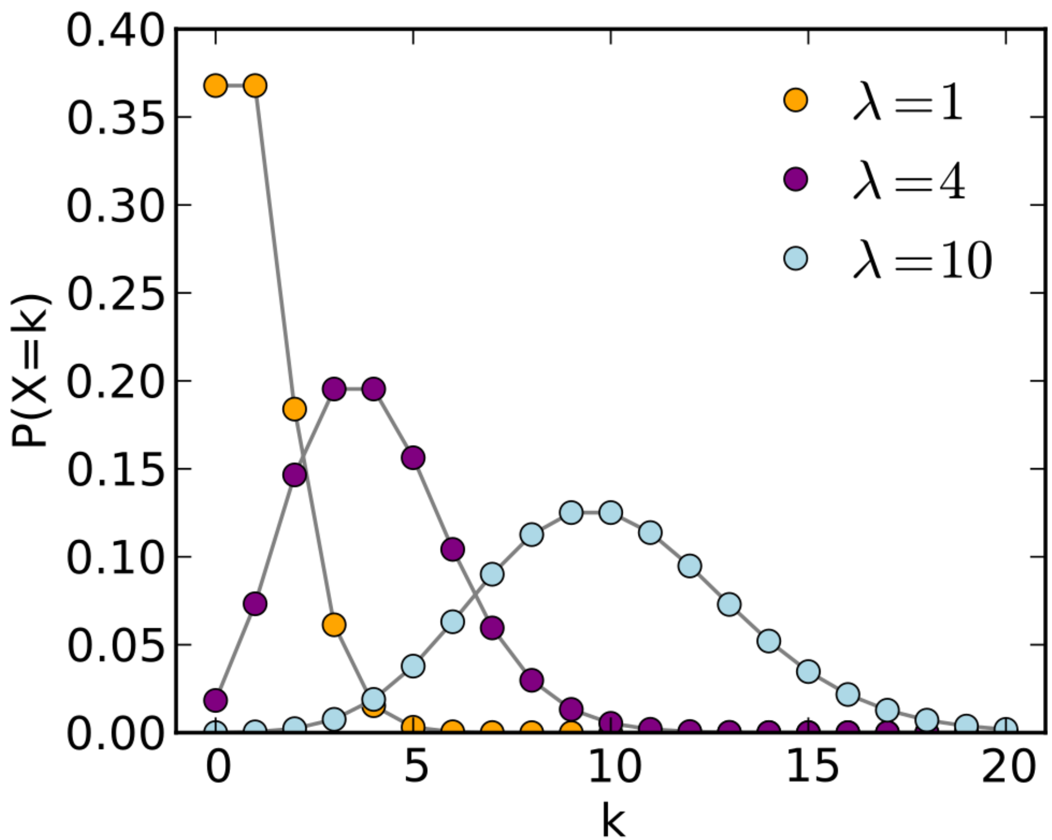
If a call center receives 4 calls per hour on average.

What is intensity **λ** here for an hour?

A. 1

B. 4

C. 8



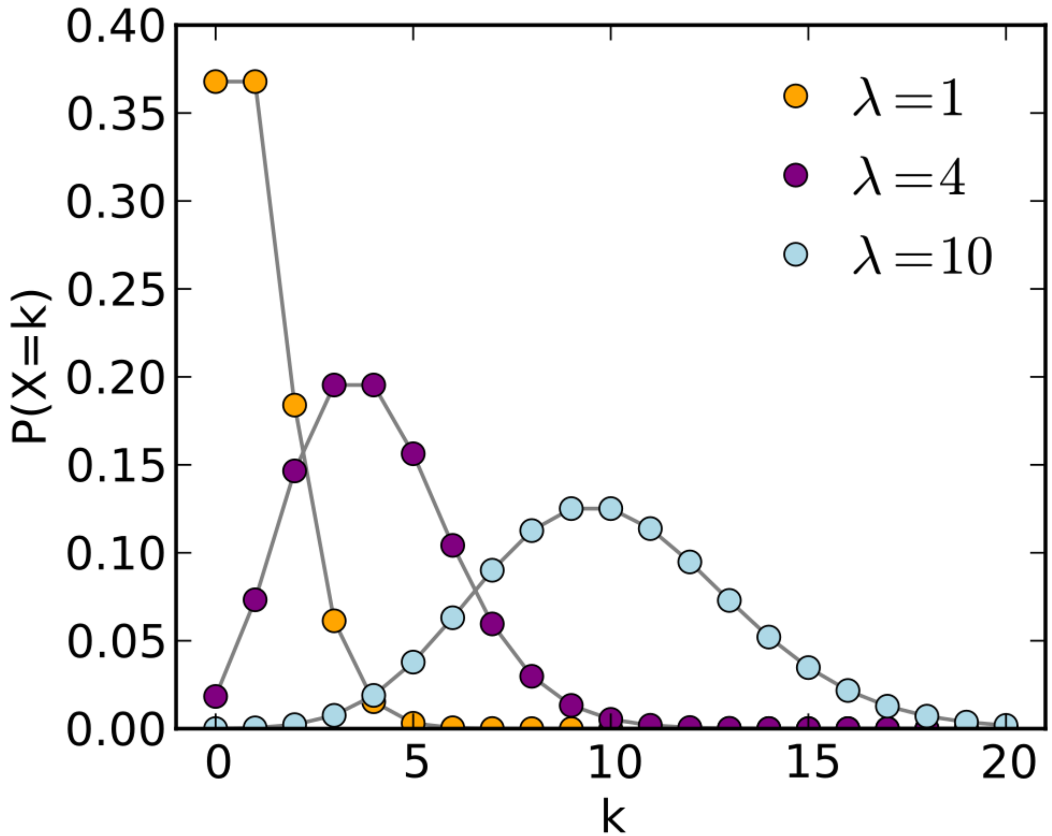
Credit: wikipedia

Q. Poisson Distribution: call center

If a call center receives 4 calls per hour on average.

What is probability the center receives 0 calls in an hour?

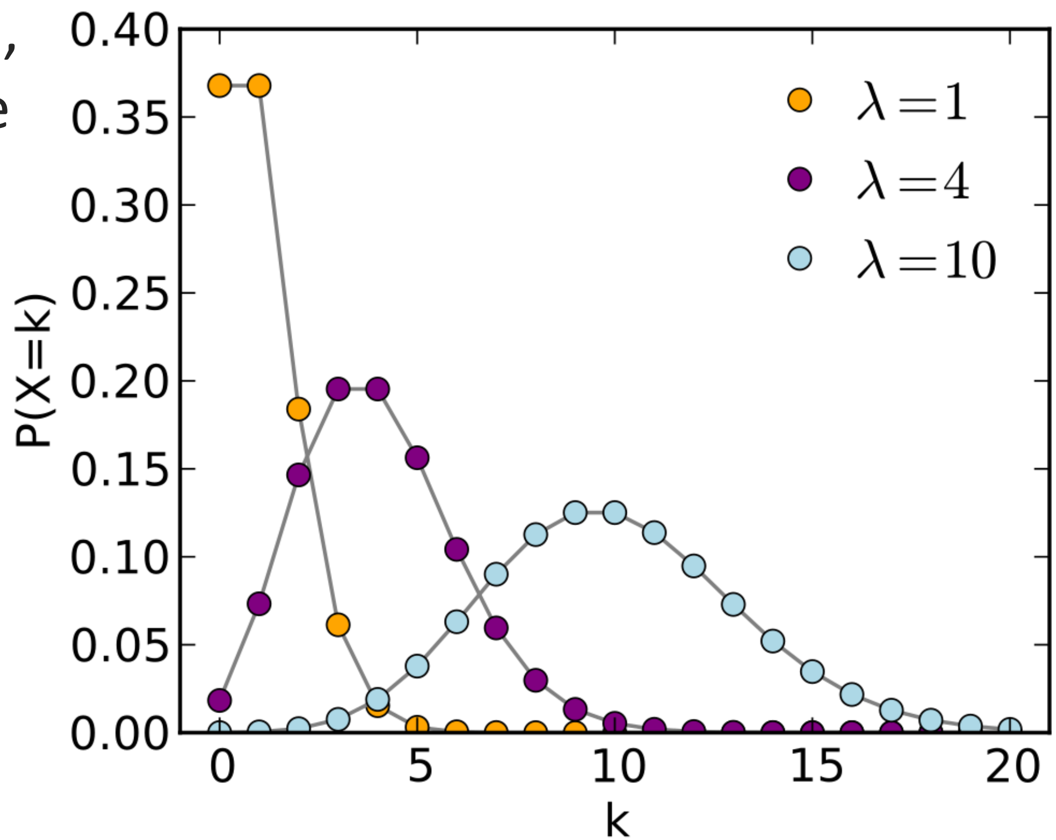
A. e^{-4}
B. 0.5
C. 0.05



Credit: wikipedia

Q. Poisson Distribution: call center

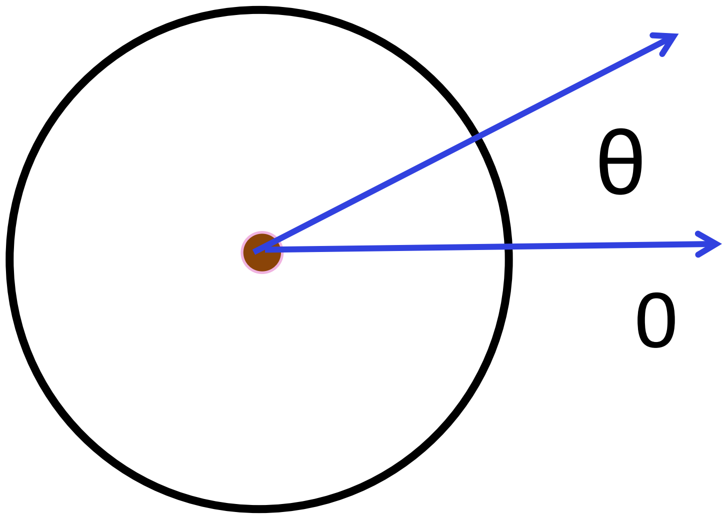
- Given a call center receives 10 calls per hour on average, what is the intensity **λ** of the distribution for calls in **Two** hours?



Credit: wikipedia

Example of a continuous random variable

- The spinner



$$\theta \in (0, 2\pi]$$

- The sample space for all outcomes is not countable

Probability density function (pdf)

- For a continuous random variable X, the probability that X=x is essentially zero for all (or most) x, so we can't define $P(X = x)$
- Instead, we define the **probability density function** (pdf) over an infinitesimally small interval dx, $p(x)dx = P(X \in [x, x + dx])$
- For $a < b$ $\int_a^b p(x)dx = P(X \in [a, b])$

Properties of the probability density function

- $p(x)$ **resembles** the probability function of discrete random variables in that
  - $p(x) \geq 0$ for all x
  - The probability of X taking all possible values is 1.

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

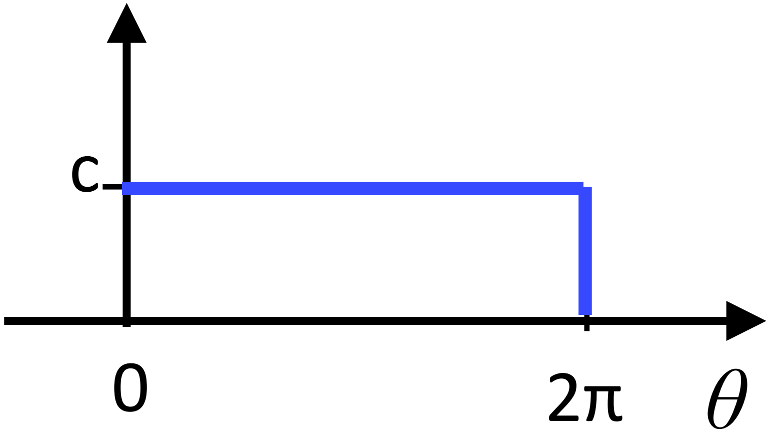
Properties of the probability density function

- $p(x)$ **differs** from the probability distribution function for a discrete random variable in that
  - $p(x)$ is not the probability that $X = x$
  - $p(x)$ can exceed 1

Probability density function: spinner

- Suppose the spinner has equal chance stopping at any position. What's the pdf of the angle θ of the spin position?

$$p(\theta) = \begin{cases} c & if\ \theta \in (0, 2\pi] \\ 0 & otherwise \end{cases}$$



- For this function to be a pdf,

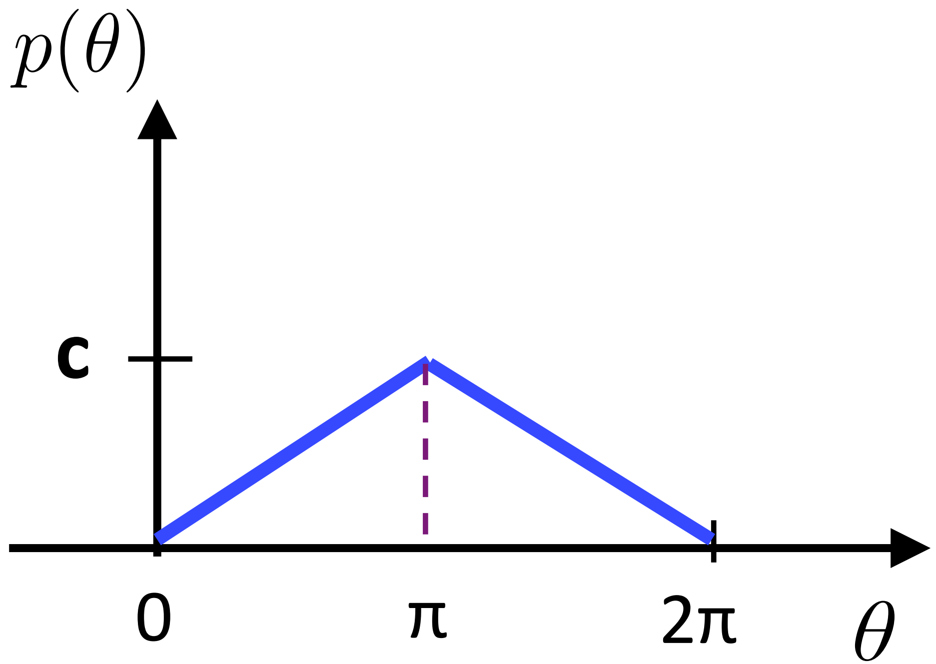
Then $$\int_{-\infty}^{\infty} p(\theta) d\theta = 1$$

Probability density function: spinner

- What the probability that the spin angle θ is within [$\frac{\pi}{12}, \frac{\pi}{7}$]?

Q: Probability density function: spinner

- What is the constant **c** given the spin angle θ has the following pdf?



A. 1
B. 1/π
C. 2/π
D. 4/π
E. 1/2π

Expectation of continuous variables

- Expected value of a continuous random variable X

weight

$$E[X] = \int_{-\infty}^{\infty} x \boxed{p(x)dx}$$

- Expected value of function of continuous random variable $Y = f(X)$

$$E[Y] = E[f(X)] = \int_{-\infty}^{\infty} f(x)p(x)dx$$

Probability density function: spinner

- Given the probability density of the spin angle θ

$$p(\theta) = \begin{cases} \frac{1}{2\pi} & if\ \theta \in (0, 2\pi] \\ 0 & otherwise \end{cases}$$

- The expected value of spin angle is

$$E[\theta] = \int_{-\infty}^{\infty} \theta p(\theta) d\theta$$

Properties of expectation of continuous random variables

- The linearity of expected value is true for continuous random variables.

$$\sum \longrightarrow \int$$

- And the other properties that we derived for variance and covariance also hold for continuous random variable

Q.

- Suppose a continuous variable has pdf

$$p(x) = \begin{cases} 2(1-x) & x \in [0,1] \\ 0 & otherwise \end{cases}$$

What is E[X]?

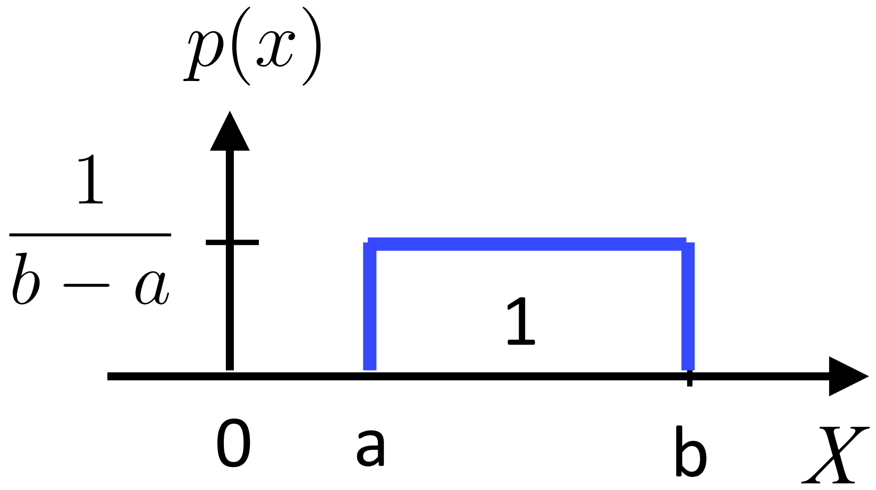
A. 1/2 B. 1/3 C. 1/4

D. 1 E. 2/3

$$E[X] = \int_{-\infty}^{\infty} xp(x)dx$$

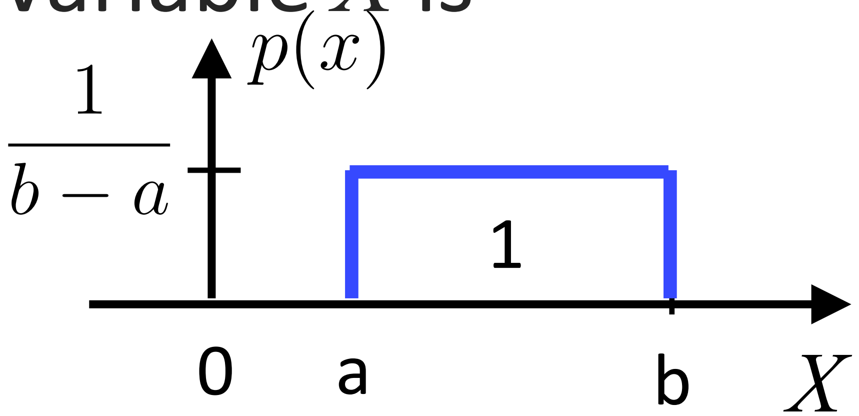
Continuous uniform distribution

- A continuous random variable X is uniform if



Continuous uniform distribution

- A continuous random variable X is uniform if

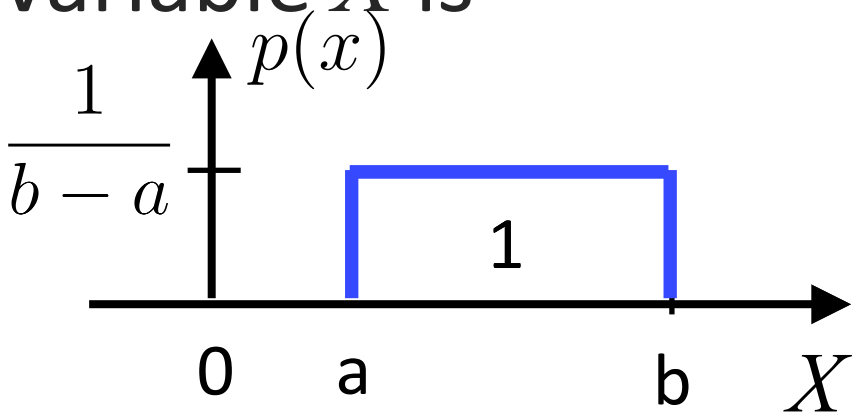


$$p(x) = \begin{cases} \frac{1}{b-a} & for\ x \in [a, b] \\ 0 & otherwise \end{cases}$$

$$E[X] = \frac{a+b}{2} \quad \& \quad var[X] = \frac{(b-a)^2}{12}$$

Continuous uniform distribution

- A continuous random variable X is uniform if



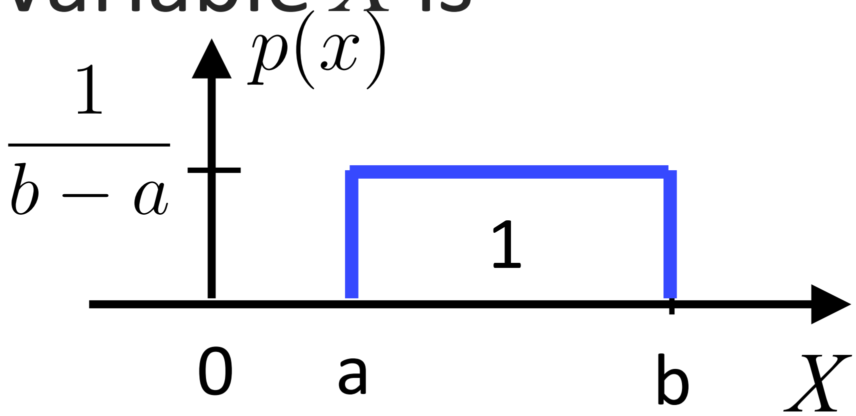
$$p(x) = \begin{cases} \frac{1}{b-a} & for\ x \in [a, b] \\ 0 & otherwise \end{cases}$$

$$E[X] = \frac{a+b}{2} \quad \& \quad var[X] = \frac{(b-a)^2}{12}$$

- Examples: 1) A dart's position thrown on the target

Continuous uniform distribution

- A continuous random variable X is uniform if



$$p(x) = \begin{cases} \frac{1}{b-a} & for\ x \in [a,b] \\ 0 & otherwise \end{cases}$$

$$E[X] = \frac{a+b}{2} \quad \& \quad var[X] = \frac{(b-a)^2}{12}$$

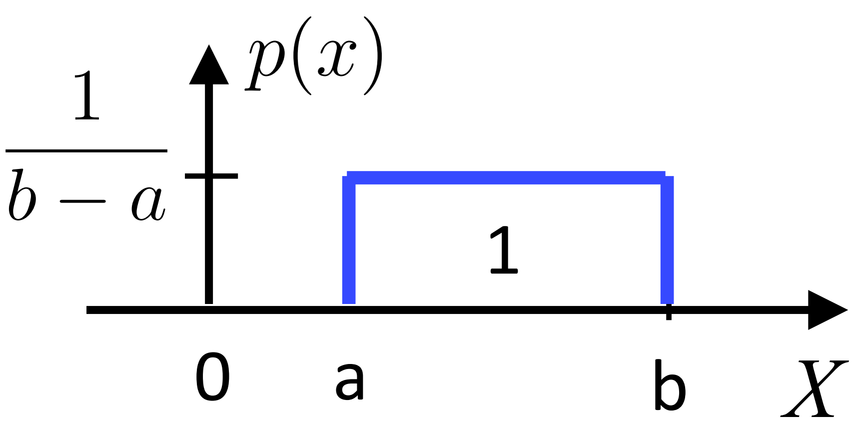
- Examples: 1) A dart's position thrown on the target 2) Often associated with **random sampling**

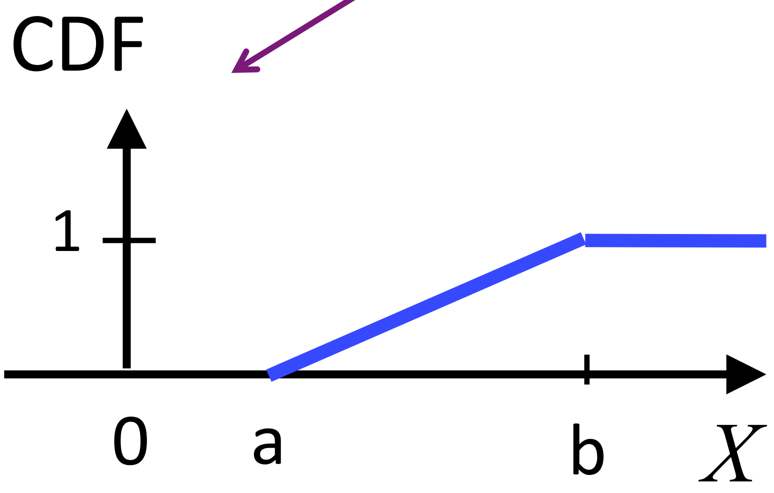
Cumulative distribution of continuous uniform distribution

- Cumulative distribution function (CDF)

$$P(X \leq x) = \int_{-\infty}^{x} p(x)dx$$

of a uniform random variable X is:

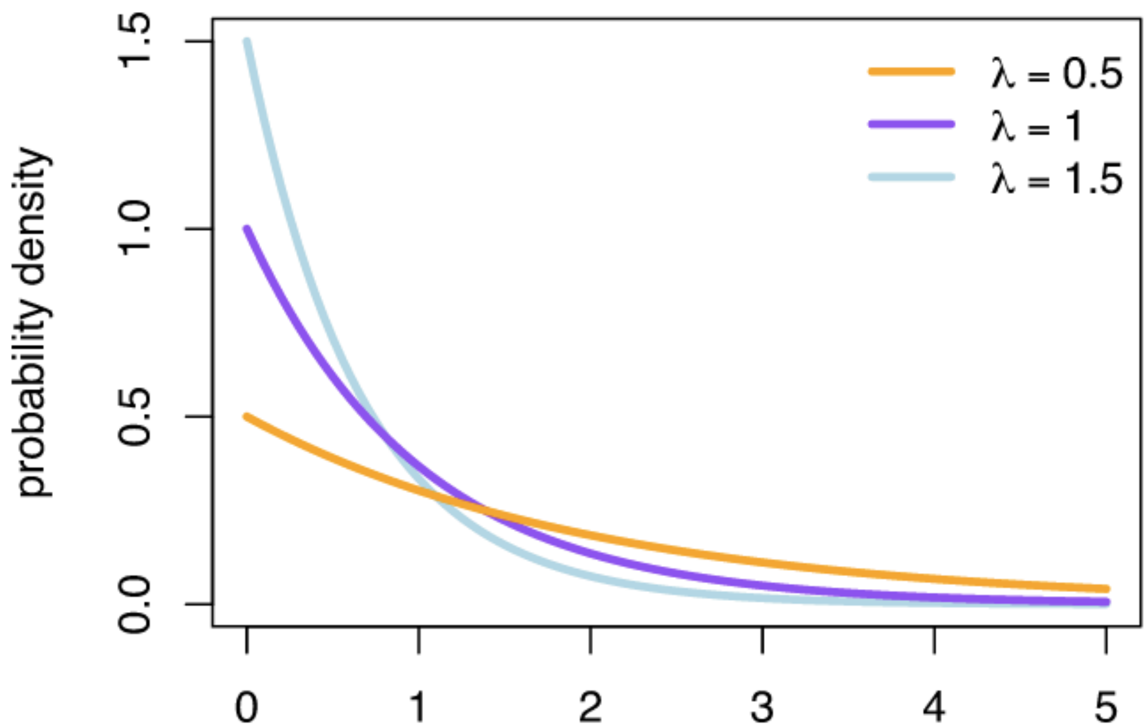




Exponential distribution

- Common Model for waiting time
- Associated with the Poisson distribution with the same **λ**

$$p(x) = \lambda e^{-\lambda x} \quad for\ x \geq 0$$



Credit: wikipedia

Exponential distribution

- A continuous random variable X is exponential if it represent the "time" until next incident in a Poisson distribution with intensity **λ**. Proof See Morris et al Pg 324.

$$p(x) = \lambda e^{-\lambda x} \quad for \ x \geq 0$$

- It's **similar** to **Geometric distribution** – the discrete version of waiting in queue

- Both are memory-less. See Degroot et al Pg 322

Exponential distribution

- A continuous random variable X is exponential if it represent the "time" until next incident in a Poisson distribution with intensity **λ**. Proof See Morris et al Pg 324.

$$p(x) = \lambda e^{-\lambda x} \quad for\ x \geq 0$$

- It's **similar** to **Geometric distribution** – the discrete version of waiting in queue

Expectations of Exponential distribution

- A continuous random variable X is exponential if it represent the "time" until next incident in a Poisson distribution with intensity **λ**.

$$p(x) = \lambda e^{-\lambda x} \quad for \ x \geq 0$$

$$\boxed{E[X] = \frac{1}{\lambda} \ \ \& \ \ var[X] = \frac{1}{\lambda^2}}$$

Example of exponential distribution

- How long will it take until the next call to be received by a call center? Suppose it's a random variable **T**. If the number of incoming call is a Poisson distribution with intensity **λ = 20 in an hour**. What is the expected time for T?

Q:

- A store has a number of customers coming on Sat. that can be modeled as a Poisson distribution. In order to measure the average rate of customers in the day, the staff recorded the time between the arrival of customers, can he reach the same goal?

    A. Yes B. No

Normal (Gaussian) distribution

- The most famous continuous random variable distribution. The probability density is this:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$E[X] = \mu \ \ \& \ \ var[X] = \sigma^2$$

Carl F. Gauss
(1777-1855)
Credit: wikipedia

Normal (Gaussian) distribution

- The most famous continuous random variable distribution.

?

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$E[X] = \mu \quad \& \quad var[X] = \sigma^2$$

Carl F. Gauss
(1777-1855)
Credit: wikipedia

Normal (Gaussian) distribution

- The most famous continuous random variable distribution.

?

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$\int_{-\infty}^{+\infty} p(x)dx = 1$$

$$E[X] = \mu \quad \& \quad var[X] = \sigma^2$$

Carl F. Gauss
(1777-1855)
Credit: wikipedia

Normal (Gaussian) distribution

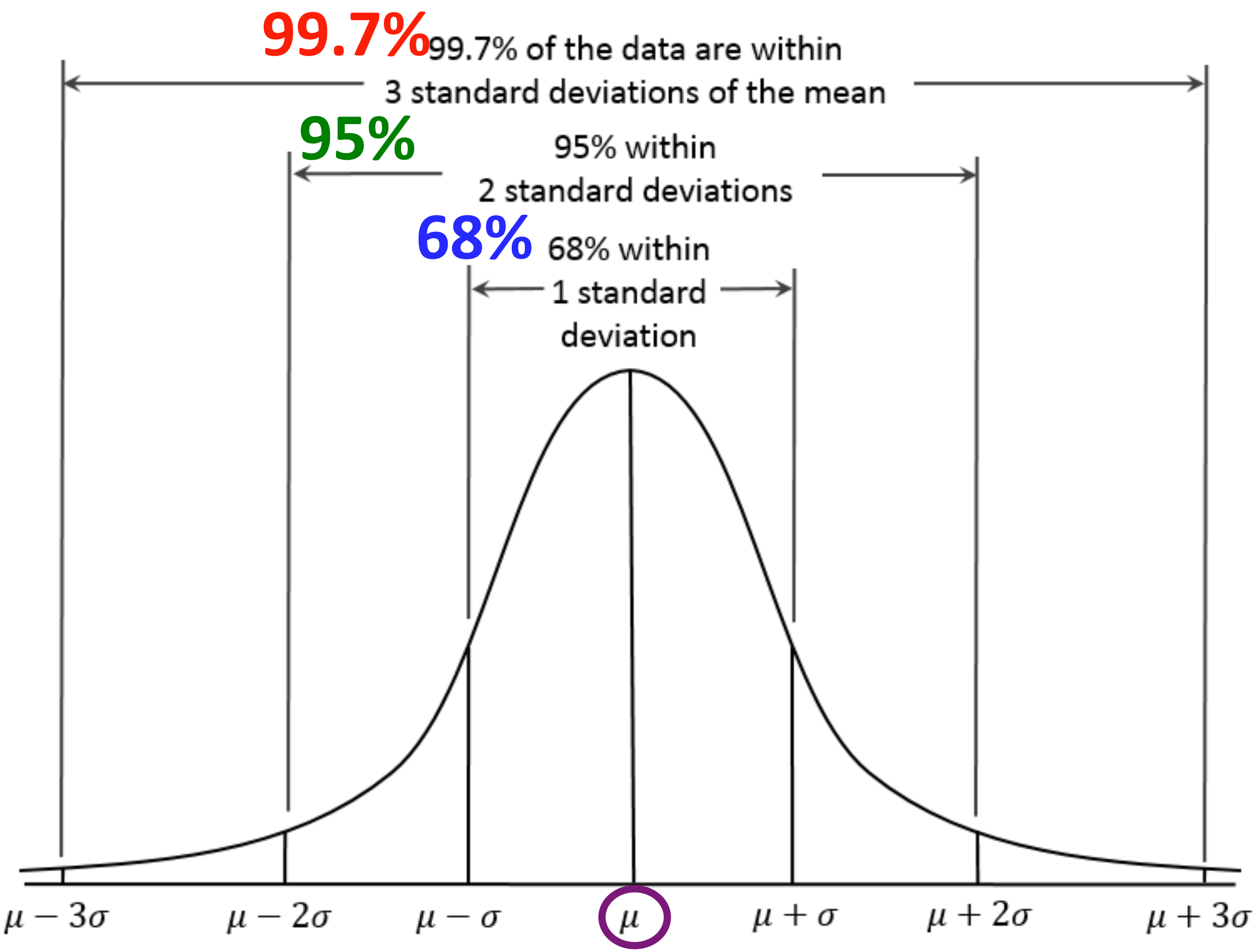
- A lot of data in nature are approximately normally distributed, ie. **Adult height**, etc.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$$E[X] = \mu \quad \& \quad var[X] = \sigma^2$$

Carl F. Gauss
(1777-1855)
Credit: wikipedia

Spread of normal (Gaussian) distributed data



Credit:
wikipedia

Standard normal distribution

- If we standardize the normal distribution (by subtracting μ and dividing by σ), we get a random variable that has standard normal distribution.

- A continuous random variable X is **standard normal** if

$$p(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$$

Derivation of standard normal distribution

$$\int_{-\infty}^{+\infty} p(x)\ dx$$

$$\hat{x} = \frac{x-\mu}{\sigma}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})\ dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} exp(-\frac{\hat{x}^2}{2})\cancel{\sigma}\ d\hat{x}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} exp(-\frac{\hat{x}^2}{2})\ d\hat{x}$$

$$= \int_{-\infty}^{+\infty} p(\hat{x})\ dx$$

Call this standard and omit using a **hat**

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})}$$

Q. What is the mean of standard normal?

A. 0

B. 1

Q. What is the standard deviation of standard normal?

A. 0

B. 1

Standard normal distribution

- If we standardize the normal distribution (by subtracting μ and dividing by σ), we get a random variable that has standard normal distribution.

- A continuous random variable X is **standard normal** if

$$p(x) = \frac{1}{\sqrt{2\pi}} exp(-\frac{x^2}{2})$$

$$\boxed{E[X] = 0 \quad \& \quad var[X] = 1}$$

Another way to check the spread of normal distributed data

- Fraction of **normal** data within **1** standard deviation from the mean.

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} exp(-\frac{x^2}{2})dx \simeq 0.68$$

- Fraction of **normal** data within **k** standard deviations from the mean.

$$\frac{1}{\sqrt{2\pi}}\int_{-k}^{k} exp(-\frac{x^2}{2})dx$$

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

Qs for discrete distributions

Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. Given there are 70% of fuji, if I want to know what is the probability I get 7 fuji in 20 apples? What is the distribution I should use?

A. Bernoulli

B. Binomial

C. Geometric

D. Poisson

E. Uniform

Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. Given there are 70% of fuji, if I want to know what is the probability I get 7 fuji in 20 apples? What is the distribution I should use? **What is the probability**?

Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. Given there are 70% of fuji, if I want to know the probability of picking the first gala on the 7th time (I can put back after each pick). What is the distribution I should use?

A. Bernoulli

B. Binomial

C. Geometric

D. Poisson

E. Uniform

Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. Given there are 70% of fuji, if I want to know the probability of picking the first gala on the 7^{th} time (I can put back after one pick). **What's the probability**?

Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. Given there are 70% of fuji, **what's the average times of picking to get the first gala?**

See you next time

See You!