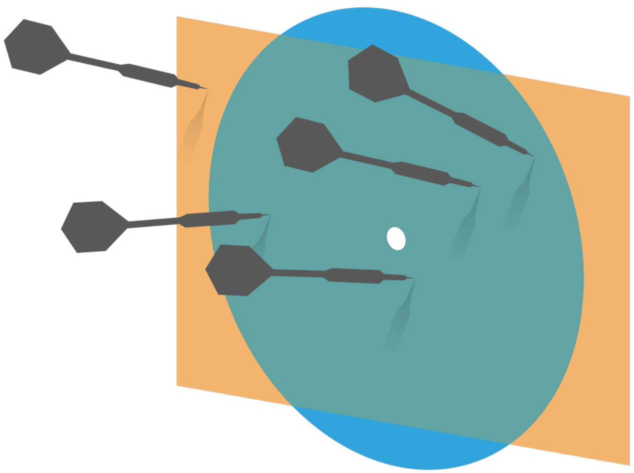
Probability and Statistics for Computer Science



Credit: wikipedia

Conditional probability comes back in matrix!

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Last Time

* Application of Clustering

  Cluster center Histogram

* Markov chain (I)

  conditional prob.

  coming back is Matrix

Objectives

Markov Chain (II)

Recap

Transition Matrix

Stationary Markov Chain

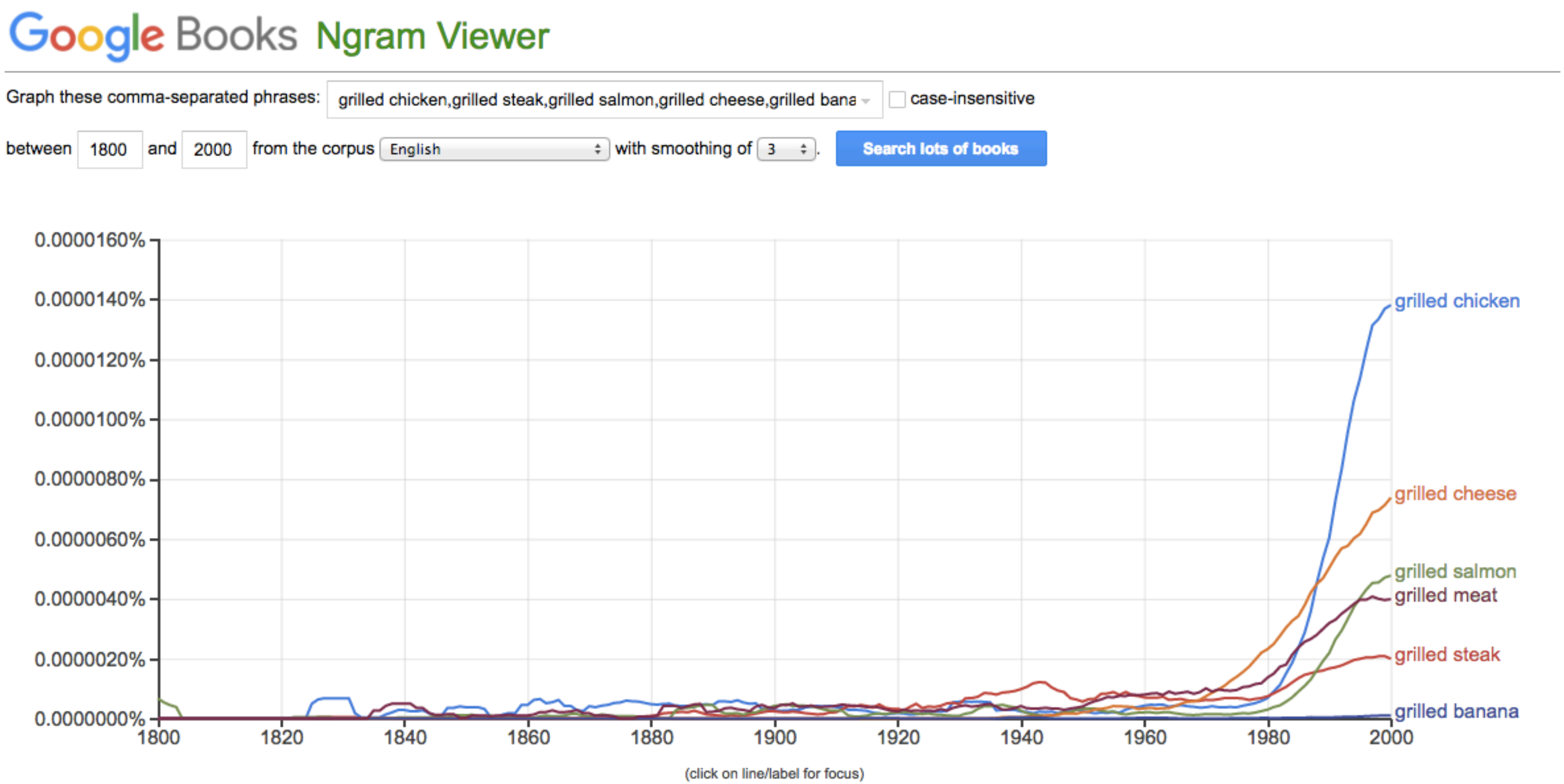
Application of Stationary Markov chain

↳ PageRank algo.

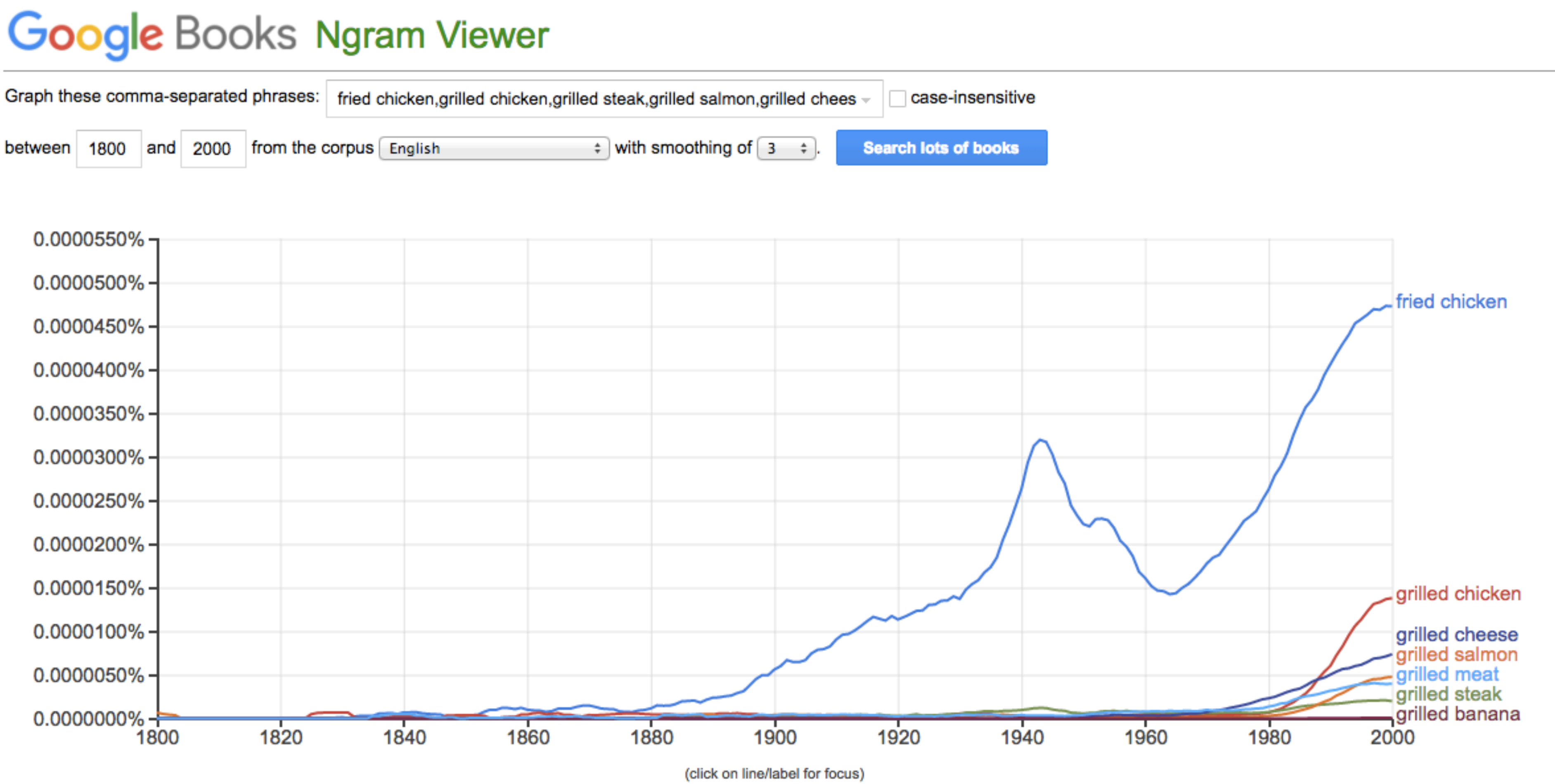
An example of dependent events in a sequence

I had a glass of wine with my grilled __________

An example of dependent events in a sequence



An example of dependent events in a sequence



Markov chain

- Markov chain is a process in which outcome of any trial in a sequence is $X_n - X_{n+1}$ **conditioned by the outcome of the trial immediately preceding, but not by earlier ones.**

X_{n+} ✓

- Such dependence is called **chain dependence**

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Andrey Markov (1856-1922)

$$p(X_{n+1} | X_n) = f(n) = P(X_{n+1} | X_n X_{n-1} \cdots X_1)$$

Markov chain in terms of probability

- Let $X_0, X_1, \ldots$ be a sequence of discrete finite-valued random variables

- The sequence is a Markov chain if the probability distribution X_t only depends on the distribution of the immediately preceding random variable X_{t-1}

$$\boxed{P(X_t|X_0 \ldots, X_{t-1}) = P(X_t|X_{t-1})}$$

- If the conditional probabilities (transition probabilities) do **NOT change with time**, it's called **constant Markov chain**.

$$P(X_t|X_{t-1}) = P(X_{t-1}|X_{t-2}) = \ldots = P(X_1|X_0)$$

$= f(t) = c$

Coin example

- Toss a fair coin until you see two heads in a row and then stop, what is the probability of stopping after exactly **n** flips?

Random variable

$\underbrace{* \; * * H H}_{n}$

X_n

Geometric

T T T . . . H

$P(n = n_0) = ?$

allele

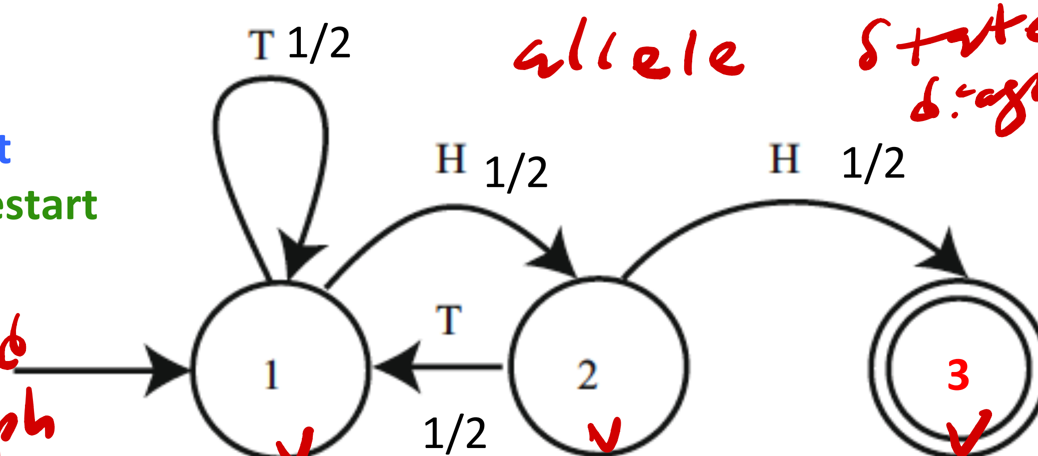
state diagram

1 -> Start or just had tail/restart

2 -> had one head after start/restart

3 -> 2heads in a row/Stop

directed graph



$N =$	1	2	3	4	5	6
Trials	T	T	H	T	H	H
$\boxed{X_N} =$	X_1	X_2	X_3	X_4	X_5	X_6
State	1	1	2	1	2	3

Markov property:

Given the current state the past doesn't matter

$$p_{ij} = P(X_{n+1}=j \mid X_n=i)$$

$$= P(X_{n+1}=j \mid X_n=i, \boxed{X_{n-1}=? \ - \ \cdots \ X_0=?})$$

This part can be any!!

The model helps form recurrence formula

- Let p_n be the probability of stopping after **n** flips

$$p_1 = 0 \quad p_2 = 1/4 \quad p_3 = 1/8 \quad p_4 = 1/8 \quad \ldots$$

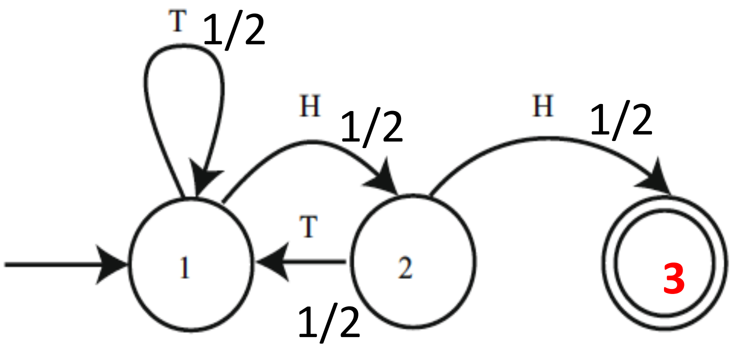
HH

*HH
THH

**HH
TTHH
HTHH

$P(n=n_0) =$

$n \uparrow$



The model helps form recurrence formula

- Let p_n be the probability of stopping after **n** flips

$$p_1 = 0 \quad p_2 = 1/4 \quad p_3 = 1/8 \quad p_4 = 1/8 \quad \dots$$

n

- If $n > 2$, there are two ways the sequence starts
  - Toss T and finish in n-1 tosses
  - Or toss HT and finish in n-2 tosses

T n-1 n

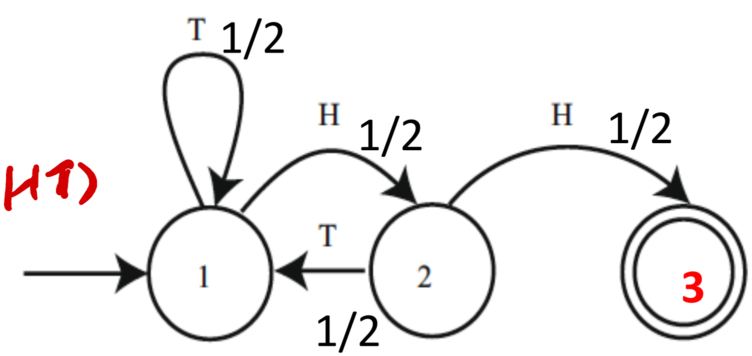
HT n-2

- So we can derive a recurrence relation

$$p_n = \frac{1}{2} p_{n-1} + \frac{1}{4} p_{n-2}$$

P(n-1|T) P(n-2|HT)

P(T) P(HT)



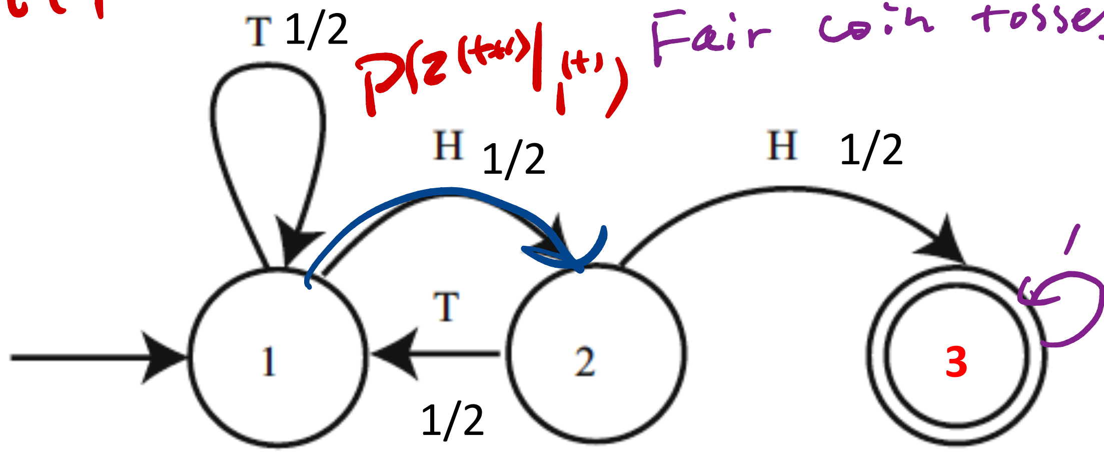
Transition probability btw states

States	1	2	3
1	$\frac{1}{2}$	$\frac{1}{2}$	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$
3	0	0	1

t, t+1

$P(z^{(t+1)} | ^{(t)})$

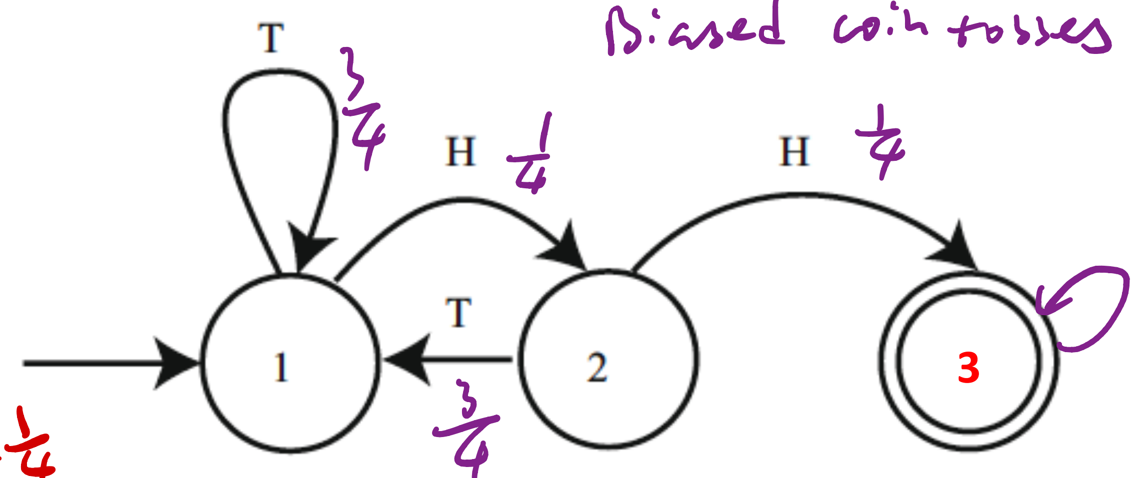
Fair coin tosses



	1	2	3
1	? $\frac{3}{4}$	$\frac{1}{4}$	0
2	$\frac{3}{4}$	0	$\frac{1}{4}$
3	0	0	1

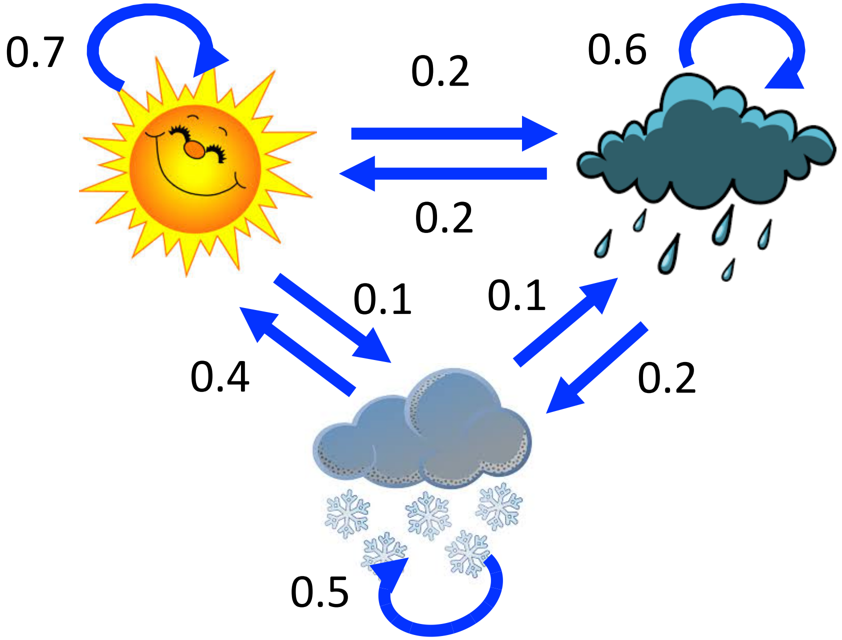
$P(2|1) = \frac{1}{4}$

Biased coin tosses



Transition probability matrix: weather model

- Let’s model daily weather as one of the three states (Sunny, Rainy, and Snowy) with Markov chain that has the transition probabilities as shown here.



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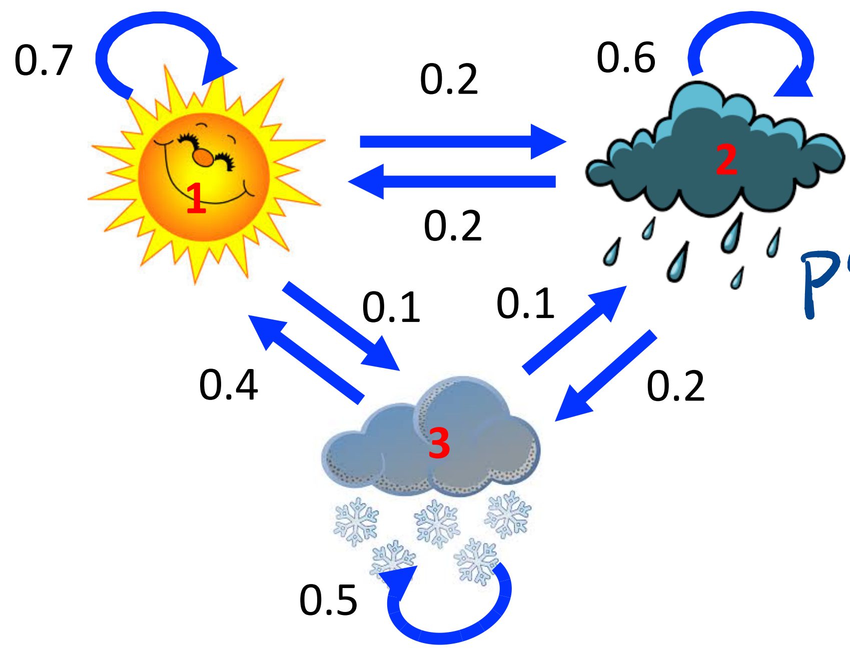


stochastic matrix

i, the current state at time point t

(t+1) **j**, the next state at time point t+1

$p(\text{sunny}^{(t+1)} \mid \text{sunny}^{(t)})$

$$\boldsymbol{P} = \begin{array}{c} \\ \\ \\ \end{array}
\begin{array}{cccc}
 & \overset{t+1}{\text{Sunny}} & \text{Rainy} & \text{Snowy} \\
 & & & \\
\end{array}$$

$$\boldsymbol{P} = \begin{array}{cc}
\begin{array}{ccc} \text{Sunny} & \text{Rainy} & \text{Snowy} \end{array} & \\
\begin{bmatrix} 0.7 & \boxed{0.2} & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} & \begin{array}{l} \overset{t}{\text{Sunny}} \\ \text{Rainy} \\ \text{Snowy} \end{array}
\end{array}$$

$P_{ij} \geq 0$

The transition probability matrix

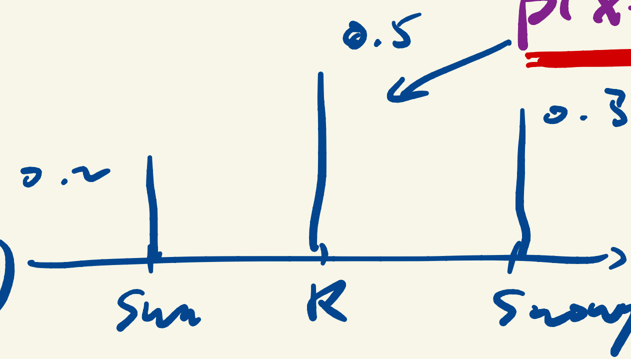
$\boxed{\text{Is this true?}}$

For a constant Markov chain, at any step t, the probability distribution among the states remain the same. $\boxed{P(X_t)}$

A. True

B. False

$P(X_t \mid X_{t-1})$



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B)\,P(A \mid B)$$

Q: The transition probabilities for a node sum to 1

A. Yes.

B. No.

Transition probability matrix properties

- The transition probability matrix $\boldsymbol{P}$ is a square matrix with entries p_{ij}
- Since $p_{ij} = P(X_t = j | X_{t-1} = i)$

$$p_{ij} \geq 0 \quad \text{and} \quad \sum_j p_{ij} = 1$$

Stochastic matrix

$X_1 \quad X_2 \quad \cdots \quad X_n$

$p(X_n = i)$

$p(X_{n+k} = j)$?

$$\boldsymbol{P} = \begin{array}{c} \begin{array}{ccc} \text{Sunny} & \text{Rainy} & \text{Snowy} \end{array} \\ \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} \text{Sunny} \\ \text{Rainy} \\ \text{Snowy} \end{array}$$

The transition probability matrix

Probability distributions over states

- Let $\boldsymbol{\pi}$ be a row vector containing the probability distribution over all the finite discrete states at t=0

3 states

$$\pi_i = P(X_0 = i)$$

Sunny Rainy Snowy

- For example: if it is rainy today, and today is t=0, then

$$\begin{array}{ccc} 1 & 2 & 3 \\ \end{array}$$

$$\boldsymbol{\pi} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

prior

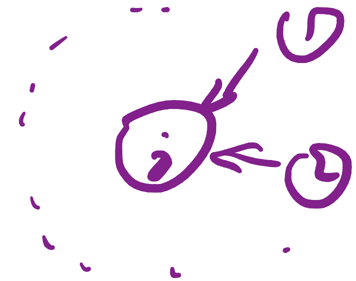
- Let $\mathbf{P}^{(t)}$ be a row vector containing the probability distribution over states at time point t

$$\mathrm{p}_i^{(t)} = P(X_t = i) \qquad \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \begin{bmatrix} ? & ? & ? \end{bmatrix}$$

$P^{(1)}$

Propagating the probability distribution

- Propagating from t=0 to t=1,



$$
\begin{aligned}
P_j^{(1)} &= P(X_1 = \boxed{j}) \\
&= \sum_i P(X_1 = j, X_0 = i) \\
&= \sum_i P(X_1 = j | X_0 = i) P(X_0 = i) \\
&= \sum_i p_{ij} \pi_i
\end{aligned}
$$

- In matrix notation,

$$\boxed{\boldsymbol{p}^{(1)} = \boldsymbol{\pi} P}$$

prior

conditional → transition matrix

t ? $p^{(t)} = p^{(t-1)} P$

Probability distributions:

- Suppose that it is rainy, we have the initial probability distribution. $\boldsymbol{\pi} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

- What are the probability distributions for tomorrow and the day after tomorrow?

$$\boldsymbol{p}^{(1)} = \boldsymbol{\pi} P$$

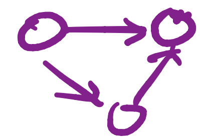
$$\boldsymbol{p}^{(2)} = \boldsymbol{p}^{(1)} P$$

Propagating to t= ∞

- We have just seen that

$$\boldsymbol{p}^{(2)} = \boldsymbol{p}^{(1)} P = (\boldsymbol{\pi} P) P = \boldsymbol{\pi} P^2$$

- So in general $\boldsymbol{p}^{(t)} = \boldsymbol{\pi} P^t$



- If one state can be reached from any other state in the graph, the Markov chain is called **irreducible** **(single chain)**.

$P^{(t)} \to S$

- Furthermore , if it satisfies: $\lim_{t \to \infty} \boldsymbol{\pi} P^t = \boldsymbol{S}$

  then the Markov chain is stationary and **S** is the stationary distribution.

Stationary distribution

- The stationary distribution $\boldsymbol{s}$ has the following property: $\boldsymbol{s}P = \boldsymbol{s}$
- $\boldsymbol{s}$ is a row eigenvector of **P** with eigenvalue 1

- In the example of the weather model, regardless of the initial distribution,

$$\boldsymbol{S} = \lim_{t \to \infty} \boldsymbol{\pi} \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}^{\mathbf{t}} = \begin{bmatrix} \frac{18}{37} & \frac{11}{37} & \frac{8}{37} \end{bmatrix}$$

> 0

Chance of being up-to-date

In a class, students are either up-to-date or behind regarding progress. If a student is up-to-date, the student has 0.8 probability remaining up-to-date, if a student is behind, the student has 0.6 probability becoming up-to-date. Suppose the course is so long that it runs life long, what is the probability any student eventually gets up-to-date?

A) 25%

$\boxed{\text{C)}}$ 75%

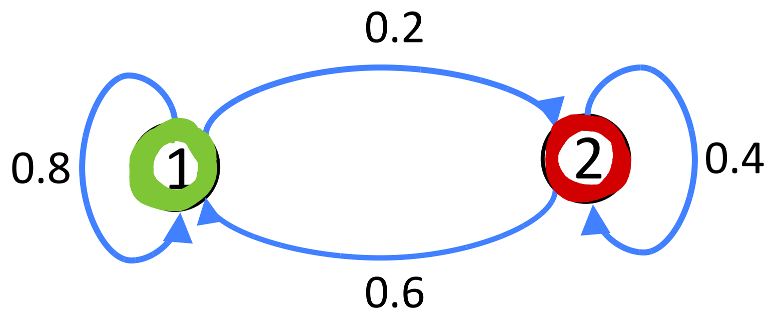
B) 50%

D) 95%

The Markov model

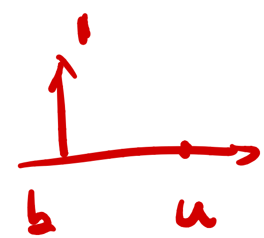
Example: Up-to-date or behind model

State 1: Up-to-date
State 2: Behind

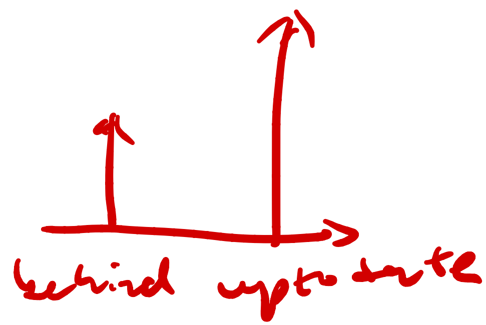


What's the transition matrix?

$$P = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$



If I start with π = [0, 1], what is my probability of being up-to-date eventually? 3/4



Solving the stationary Markov

$$SP = S$$

$$(SP)^T = S^T \Rightarrow P^T S^T = S^T$$

$$Ax = x \qquad \lambda = 1$$

$$S^T = u \qquad A = P^T$$

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}^T u = u$$

$$[A - 1]u = 0 \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

75%

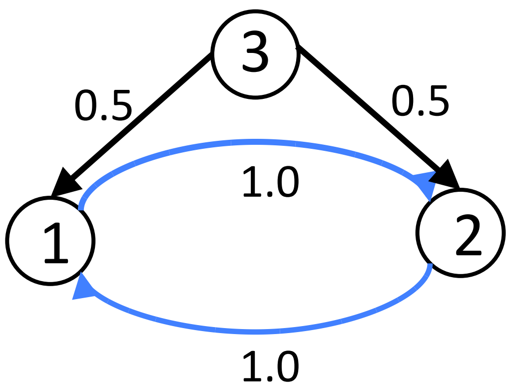
up to date!

$$\begin{bmatrix} -0.2 & 0.6 \\ 0.2 & -0.6 \end{bmatrix} u = 0$$

$$\boxed{u_1 = \frac{3}{4}}$$

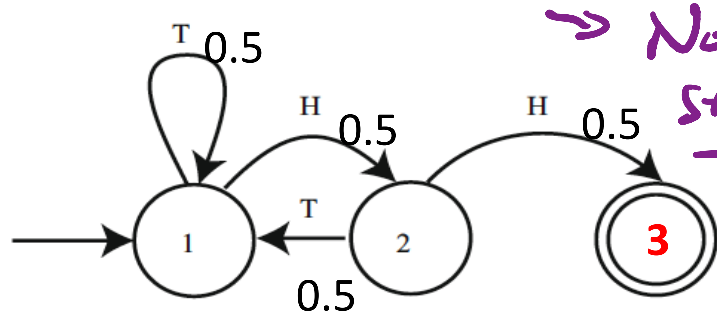
$$u_1 + u_2 = 1 \qquad u_2 = \frac{1}{4}$$

Examples of non-stationary Markov chains



Periodic

Not irreducible → Not stationary



Absorbing

$SP = S$

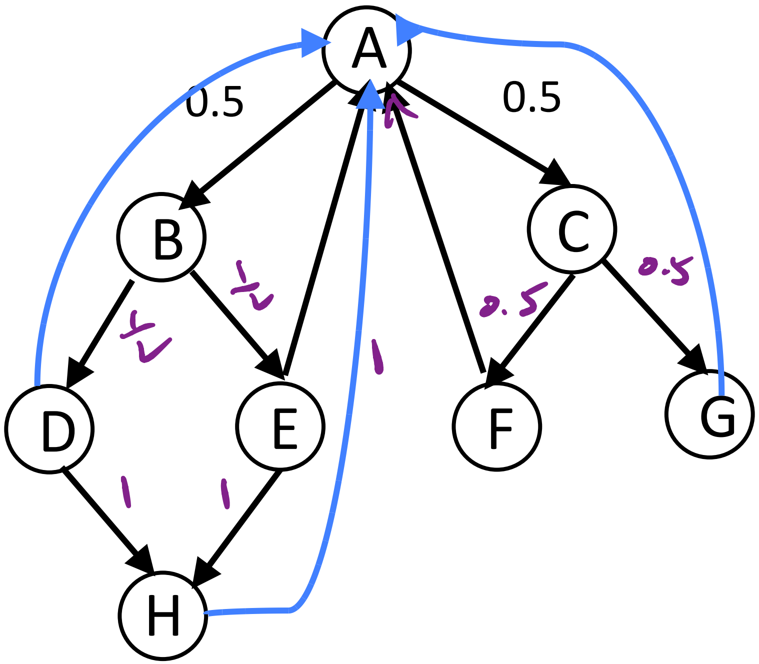
PageRank Example

- How to rate web pages objectively?
- The PageRank algorithm by Page et al. made **Google** successful
- The method utilized **Markov chain model** and applied it to the large list of webpages.
- To illustrate the point, we use a small-size example and assume a simple **stationary model**.

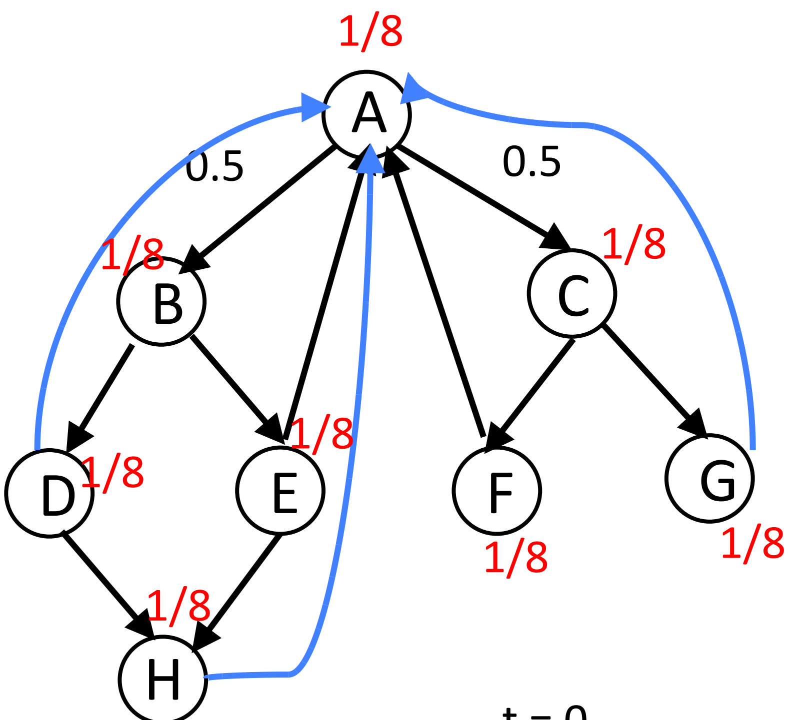
$$\lim_{t \to \infty} \pi P^t$$

Suppose we are randomly surfing a network of webpages

$\frac{1}{8}$



Initialize the distribution uniformly

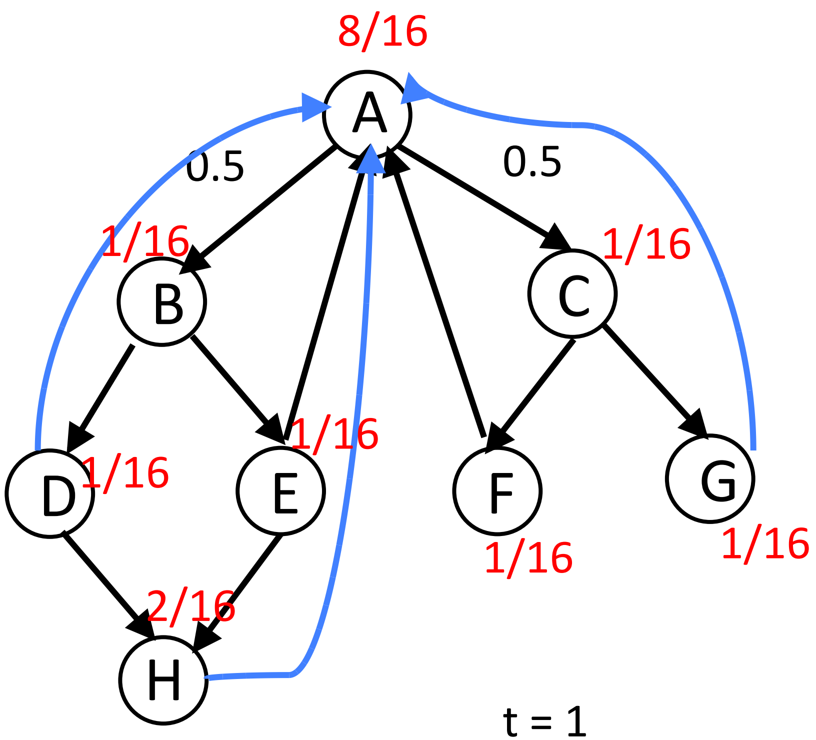


t = 0

$$\boldsymbol{\pi} = \begin{bmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \end{bmatrix}$$

$$P = \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

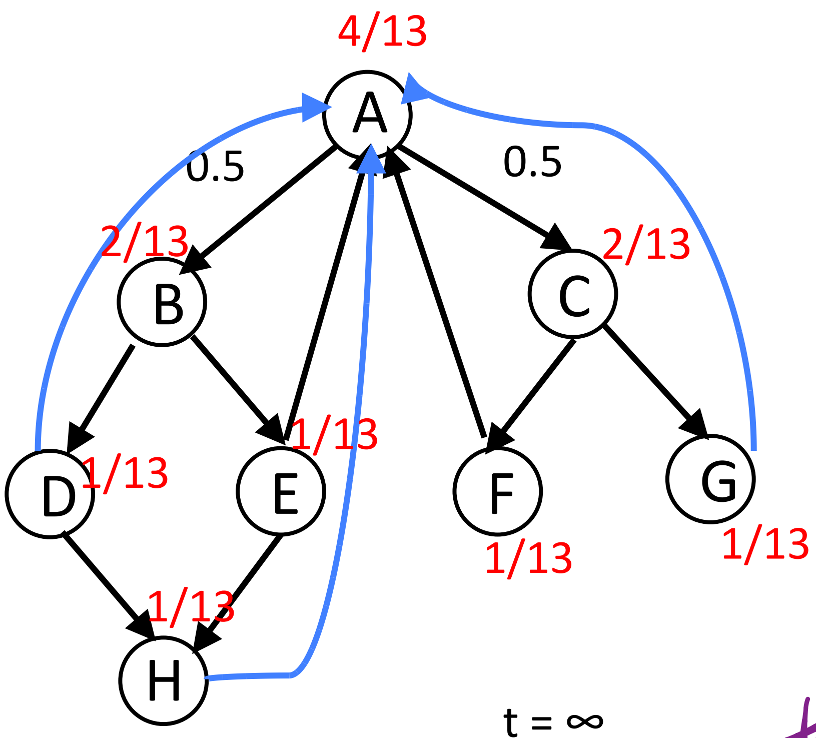
Update the distribution iteratively



$$\lim_{t \to \infty} \pi P^t = ?$$

$$\pi P^t$$

Until the stationary distribution



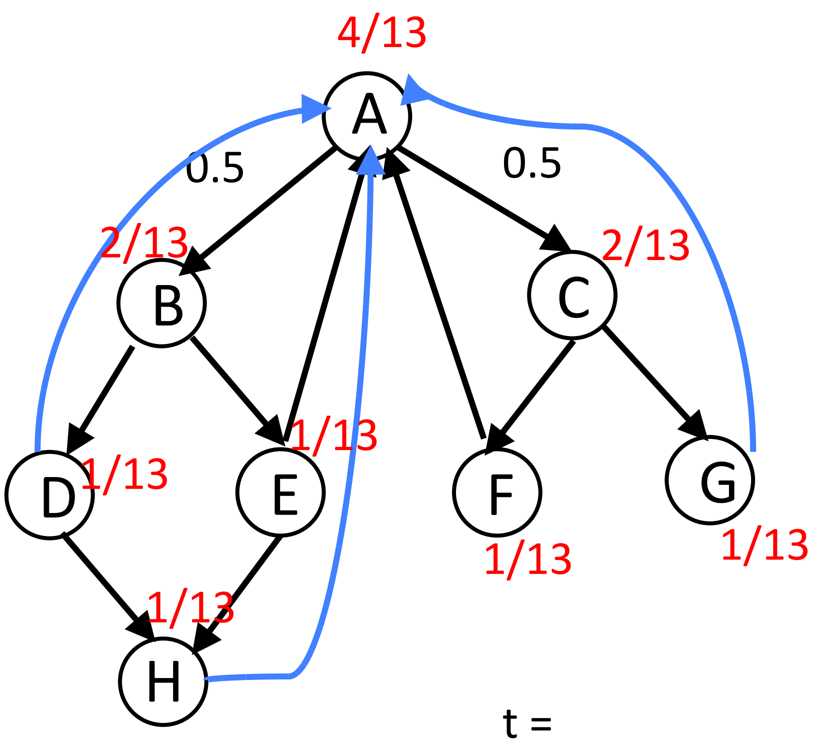
why not try $sP = s$?

page ~ 1.7×10^9

If the surfer get trapped

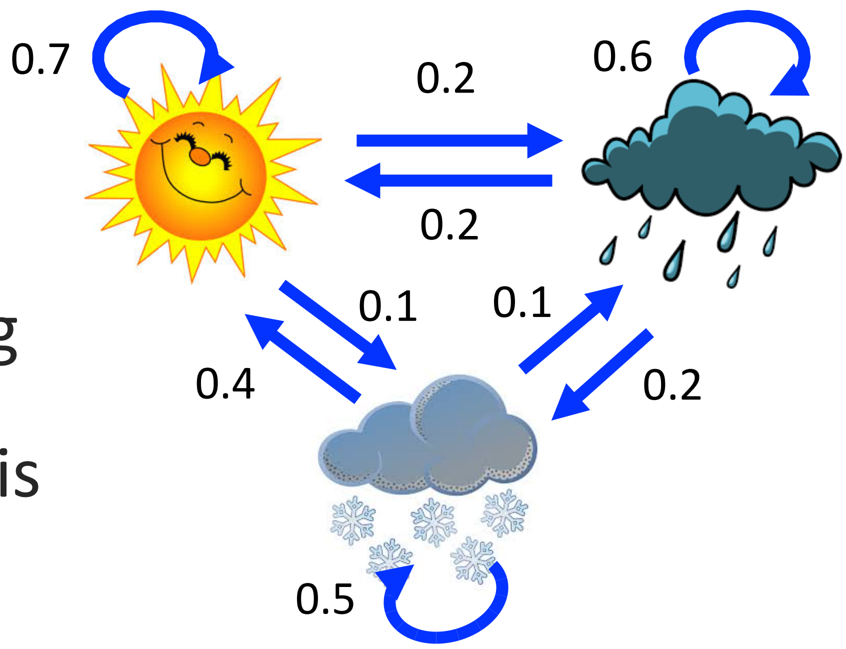
- Allow "teleport" with small probability from any page to another

- Or allow "teleport" with user input of URL



Diverse applications of Markov Model

- Communication network
- Queue modeling
- DNA sequence modeling
- Natural language processing
- Single-cell large data analysis
- Financial/Economic model
- Music



Final Exam

- Time: 1:30pm-4:30pm 12/11 Fri. Central Time
- Conflicts need to be requested 2 days ahead.
- Duration: 3hrs
- Content coverage: Ch1-14, except 8, details are on Compass
- Open book and lecture notes
- Format: 50 multiple choices

Additional References

- Robert V. Hogg, Elliot A. Tanis and Dale L. Zimmerman. “Probability and Statistical Inference”
- Kelvin Murphy, “Machine learning, A Probabilistic perspective”

Acknowledgement

Thank You!