Conditional probability comes back in matrix!

Credit: wikipedia
Last Time

- Application of Clustering
  Cluster Center Histogram

- Spectral Clustering
Objectives

Markov Chain (I)
Motivation

- So far, the processes we learned such as **Bernoulli** and **Poisson** process are sequences of **independent** trials.

- There are a lot of real world situations where sequences of events are **Not independent** in comparison.

- **Markov chain** is one type of characterization of a series of **dependent** trials.
An example of dependent events in a sequence

I had a glass of wine with my grilled ________
An example of dependent events in a sequence
An example of dependent events in a sequence
Markov chain

Markov chain is a process in which outcome of any trial in a sequence is conditioned by the outcome of the trial immediately preceding, but not by earlier ones.

Such dependence is called chain dependence

Andrey Markov (1856-1922)
Markov chain in terms of probability

Let \( X_0, X_1, \ldots \) be a sequence of discrete finite-valued random variables.

The sequence is a Markov chain if the probability distribution \( X_t \) only depends on the distribution of the immediately preceding random variable \( X_{t-1} \):

\[
P(X_t|X_0, \ldots, X_{t-1}) = P(X_t|X_{t-1})
\]

If the conditional probabilities (transition probabilities) do NOT change with time, it’s called constant Markov chain.

\[
P(X_t|X_{t-1}) = P(X_{t-1}|X_{t-2}) = \ldots = P(X_1|X_0)
\]
Toss a fair coin until you see two heads in a row and then stop, what is the probability of stopping after exactly \( n \) flips?

Use a state diagram, which is a **directed graph**. Circles are the states of likely outcomes. Arrow directions show the direction of transitions. Numbers over the arrows show transition probabilities.

1 -> Start or just had tail/restart
2 -> had one head after start/restart
3 -> 2 heads in a row/Stop
The model helps form recurrence formula

Let $p_n$ be the probability of stopping after $n$ flips.

\[
p_1 = 0 \quad p_2 = \frac{1}{4} \quad p_3 = \frac{1}{8} \quad p_4 = \frac{1}{8} \quad \ldots
\]
The model helps form recurrence formula

Let \( p_n \) be the probability of stopping after \( n \) flips

\[
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\]

If \( n > 2 \) there are two ways the sequence starts

- Toss T and finish in \( n-1 \) tosses
- Or toss HT and finish in \( n-2 \) tosses

So we can derive a recurrence relation

\[
p_n = \frac{1}{2} p_{n-1} + \frac{1}{4} p_{n-2}
\]
Transition probability btw states
Let’s model daily weather as one of the three states (Sunny, Rainy, and Snowy) with Markov chain that has the transition probabilities as shown here.
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The transition probability matrix is:

\[
P = \begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.4 & 0.1 & 0.5
\end{bmatrix}
\]

- \(i\), the current state at time point \(t\)
- \(j\), the next state at time point \(t+1\)
Q: Is this TRUE?

For a constant Markov Chain, at any step t, the probability distribution among the states remain the same.

A. Yes.

B. No.
Q: The transition probabilities for a node sum to 1

A. Yes.

B. No.

Only the row sum is 1, that is: the probabilities associated with outgoing arrows sum to 1.
The transition probability matrix $P$ is a square matrix with entries $p_{ij}$.

Since $p_{ij} = P(X_t = j | X_{t-1} = i)$,

\[ p_{ij} \geq 0 \quad \text{and} \quad \sum_j p_{ij} = 1 \]

\[
P = \begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.4 & 0.1 & 0.5
\end{bmatrix}
\]

The transition probability matrix
Let \( \mathbf{\pi} \) be a row vector containing the probability distribution over all the finite discrete states at \( t=0 \)

\[
\pi_i = P(X_0 = i)
\]

For example: if it is rainy today, and today is \( t=0 \), then

\[
\mathbf{\pi} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}
\]

Let \( \mathbf{P}^{(t)} \) be a row vector containing the probability distribution over states at time point \( t \)

\[
p_i^{(t)} = P(X_t = i)
\]
Propagating the probability distribution

**Propagating from t=0 to t=1,**

\[
P_j^{(1)} = P(X_1 = j)
= \sum_i P(X_1 = j, X_0 = i)
= \sum_i P(X_1 = j | X_0 = i) P(X_0 = i)
= \sum_i p_{ij} \pi_i
\]

**In matrix notation,**

\[
p^{(1)} = \pi P
\]
Probability distributions:

Suppose that it is rainy, we have the initial probability distribution.

\[ \pi = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \]

What are the probability distributions for tomorrow and the day after tomorrow?

\[ p^{(1)} = \pi P \]

\[ p^{(2)} = p^{(1)} P \]
We have just seen that
\[ p^{(2)} = p^{(1)} P = (\pi P) P = \pi P^2 \]
So in general
\[ p^{(t)} = \pi P^t \]
If one state can be reached from any other state in the graph, the Markov chain is called **irreducible** (single chain).
Furthermore, if it satisfies:
\[ \lim_{{t \to \infty}} \pi P^t = S \]
then the Markov chain is stationary and \( S \) is the stationary distribution.
Stationary distribution

- The stationary distribution $\mathbf{s}$ has the following property: $\mathbf{s} \mathbf{P} = \mathbf{s}$
- $\mathbf{s}$ is a row eigenvector of $\mathbf{P}$ with eigenvalue 1
- In the example of the weather model, regardless of the initial distribution, 

\[
\mathbf{S} = \lim_{t \to \infty} \pi \begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.4 & 0.1 & 0.5
\end{bmatrix}^t = \begin{bmatrix}
\frac{18}{37} & \frac{11}{37} & \frac{8}{37}
\end{bmatrix}
\]
State 1: Up-to-date
State 2: Behind

What's the transition matrix?
If I start with $\pi = [0, 1]$, what is my probability of being up-to-date eventually? $\frac{3}{4}$
Example: Up-to-date or behind model

\[ SP = S \Rightarrow (SP)^T = S^T \Rightarrow P^T S^T = S^T \]
\[ (P^T - I)S^T = 0 \]
Examples of non-stationary Markov chains

Periodic

Absorbing

Kelvin Murphy, “Machine learning, A Probabilistic perspective”
See you next time

See You!