“A major use of probability in statistical inference is the updating of probabilities when certain events are observed” – Prof. M.H. DeGroot

Credit: wikipedia
Laws of Sets

Commutative Laws

\[ A \cap B = B \cap A \]
\[ A \cup B = B \cup A \]

Associative Laws

\[(A \cap B) \cap C = A \cap (B \cap C)\]
\[(A \cup B) \cup C = A \cup (B \cup C)\]

Distributive Laws

\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
## Laws of Sets

### Idempotent Laws

- \( A \cap A = A \)
- \( A \cup A = A \)

### Identity Laws

- \( A \cup \emptyset = A \)
- \( A \cap U = A \)

### Complement Laws

- \( A \cup A^c = U \)
- \( A \cap A^c = \emptyset \)
- \( U^c = \emptyset \)
- \( \emptyset^c = U \)

### Involution Law

\( (A^c)^c = A \)

### De Morgan’s Laws

- \( (A \cap B)^c = A^c \cup B^c \)
- \( (A \cup B)^c = A^c \cap B^c \)

- \( U \) is the complete set
Objectives

- Probability
  - More probability calculation
  - Conditional Probability
    - Bayes rule
  - Independence
The United States Senate contains two senators from each of the 50 states. If a committee of eight senators is selected at random, what is the probability that it will contain at least one of the two senators from IL?
Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.
Conditional Probability

Motivation of conditional probability
Conditional Probability

Example:

An insurance company knows in a population of 100 thousands females, 89.835% expect to live to age 60, while 57.062% can expect to live to 80. Given a woman at the age of 60, what is the probability that she lives to 80?
Conditional Probability

Given the condition she is 60 already, the size of the sample space for the outcomes has changed to

89,835 instead of 100,000
Conditional Probability

The probability of $A$ given $B$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(B) \neq 0$

The “Size” analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom
Conditional Probability

\( A \): a woman lives to 80

\( B \): a woman is at 60 now

\[ P(A|B) = \frac{57,062}{89,835} = 0.6352 \]

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

While \( P(A) = \frac{57,062}{100,000} = 0.57062 \)
Conditional Probability: die example

Throw 5-sided fair die twice.

\[ A : \max(X, Y) = 4 \]
\[ B : \min(X, Y) = 2 \]

\[ P(A|B) = ? \]
Conditional probability, that is?

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0 \]
Multiplication rule using conditional probability

Joint event

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0 \]

\[ \Rightarrow P(A \cap B) = P(A|B)P(B) \]
Multiplication using conditional probability

\[ P(A \cap B) = P(A|B)P(B) \]

\[ P(soup \cap meat) = P(meat|soup)P(soup) = 0.5 \times 0.8 = 0.4 \]
Symmetry of joint event in terms of conditional prob.

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0 \]

\[ \Rightarrow P(A \cap B) = P(A|B)P(B) \]

\[ \Rightarrow P(B \cap A) = P(B|A)P(A) \]
Symmetry of joint event in terms of conditional prob.

\[ P(B \cap A) = P(A \cap B) \]

\[ \implies P(A|B)P(B) = P(B|A)P(A) \]
The famous Bayes rule

\[ P(A|B)P(B) = P(B|A)P(A) \]

\[ \Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

Thomas Bayes (1701-1761)
Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?
There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

\[
P(B|L) = \frac{P(L|B)P(B)}{P(L)}
\]
Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

\[ P(A|L) = ? \]
Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

\[ P(A|L) =? \]

\[ P(A|L) = \frac{P(L|A)P(A)}{P(L)} \]

Or in this case

\[ P(A|L) = 1 - P(B|L) \]
Given the above information, what is the probability that it came from factory A?

\[ P(A|L) = ? \]

\[ P(A|L) = \frac{P(L|A)P(A)}{P(L)} \]

Or in this case

\[ P(A|L) = 1 - P(B|L) \]
Total probability

\[ P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \]
\[ = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \]
Total probability general form

\[ P(A) = \sum_{j} (P(A|B_j)P(B_j)) \]

if \( B_i \cap B_j = \emptyset \) for all \( i \neq j \)
Bayes rule using total prob.

\[ P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)} \]

\[ = \frac{P(A|B_j)P(B_j)}{\sum_j P(A|B_j)P(B_j)} \]
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \( P(D|T) \), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is \(\frac{1}{100,000}\). If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \(P(D|T)\), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]
Independence

One definition:

\[
\begin{align*}
P(A|B) &= P(A) \text{ or } \\
P(B|A) &= P(B)
\end{align*}
\]

Whether A happened doesn’t change the probability of B and vice versa
Independence: example

Suppose that we have a fair coin and it is tossed twice. Let $A$ be the event “the first toss is a head” and $B$ the event “the two outcomes are the same.”

These two events are independent!
**Independence**

Alternative definition

\[ P(A|B) = P(A) \]

\[ \Rightarrow \frac{P(A \cap B)}{P(B)} = P(A) \]

\[ \Rightarrow P(A \cap B) = P(A)P(B) \]
Suppose you draw one card from a standard deck of cards. $E_1$ is the event that the card is a King, Queen or Jack. $E_2$ is the event the card is a Heart. Are $E_1$ and $E_2$ independent?
Simulation of Conditional Probability

http://www.randomservices.org/random/apps/ConditionalProbabilityExperiment.html
Additional References

- Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability"

- Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics"
Assignments

⊙ Work on Module Week 3 on Canvas

⊙ Next time: More on independence and conditional probability
See you next time

See You!