“A major use of probability in statistical inference is the updating of probabilities when certain events are observed” – Prof. M.H. DeGroot
Counting: how many ways?

if we put 7 hats (indistinguishable) on 7 people out of 10 people randomly?
Warm up: which is larger?

\[ P(\text{A} \cap \text{B}) \text{ or } P(\text{A} | \text{B}) \]

A) \( P(\text{A} \cap \text{B}) \)

\( \boxed{\text{B}) P(\text{A} | \text{B})} \)

C) None

\[ P(\text{A} | \text{B}) = \frac{P(\text{A} \cap \text{B})}{P(\text{B})} \]

\( P(\text{B}) < 1 \)
Last time

* More probability calculation using counting
* Conditional probability Multiplication rule
Objectives

- Conditional Probability
- Review
- Bayes rule
- Total probability
- Independence
The probability of $A$ given $B$

\[ P(A | B) = \frac{P(A \cap B)}{P(B)} \]

\[ P(B) \neq 0 \]

The line-crossed area is the new sample space for conditional $P(A | B)$
Joint Probability Calculation

\[ P(A \cap B) = P(\frac{A}{B})P(B) \]

\[ P(\text{soup} \cap \text{meat}) = P(\frac{\text{meat}}{\text{soup}})P(\text{soup}) \]

\[ = 0.5 \times 0.8 = 0.4 \]
Joint Probability Calculation

\[ P(A \cap B) = P(A \mid B) P(B) \]

\[ P(soup \cap meat) = 0.5 \times 0.8 = 0.4 \]
Joint Probability Calculation

\[ \Rightarrow P(A \cap B) = P(A|B)P(B) \]

\[ P(\text{soup} \cap \text{meat}) = P(\text{meat}|\text{soup})P(\text{soup}) = 0.5 \times 0.8 = 0.4 \]
Symmetry of joint event in terms of conditional prob.

\[ P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0 \]

\[ \Rightarrow P(A \cap B) = P(A|B)P(B) \]
\[ \Rightarrow P(B \cap A) = P(B|A)P(A) \]
Symmetry of joint event in terms of conditional prob.

\[ B \cap A = A \cap B \]

\[ P(B \cap A) = P(A \cap B) \]

\[ \Rightarrow \]

\[ P(A|B)P(B) = P(B|A)P(A) \]
The famous **Bayes rule**

\[
P(A|B)P(B) = P(B|A)P(A)
\]

\[
P(A|B) = \frac{P(B|A)P(A)}{P(B)}
\]

\[
P(A|D) = \frac{P(D|A)P(A)}{P(D)}
\]
The famous **Bayes rule**

\[ P(A|B)P(B) = P(B|A)P(A) \]

\[ \Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

Thomas Bayes (1701-1761)
Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

Let:
- $E_A$: it came from Factory A
- $E_B$: it came from Factory B
- $\bar{E}_A$: it did not come from Factory A
- $\bar{E}_B$: it did not come from Factory B
- $E$: it is a lemon

Then:
$$P(E|\bar{E}_B) = \frac{P(E|\bar{E}_B) \cdot P(\bar{E}_B)}{P(E)}$$

We want to find $P(E|\bar{E}_B)$. We can use Bayes' rule to express it in terms of the prior probabilities and the likelihoods:

$$P(E|\bar{E}_B) = \frac{P(E \cap \bar{E}_B)}{P(\bar{E}_B)}$$

Since $E$ is a lemon, $E \cap \bar{E}_B$ is the event that the car is a lemon and not from Factory A. Therefore:

$$P(E|\bar{E}_B) = \frac{P(E \cap \bar{E}_B)}{P(\bar{E}_B)} = \frac{P(E \cap \bar{E}_B)}{1 - P(E)}$$

We can calculate $P(E)$ as the total probability of getting a lemon, considering both factories:

$$P(E) = P(E|E_A) \cdot P(E_A) + P(E|\bar{E}_A) \cdot P(\bar{E}_A)$$

Since $P(E|E_A) = \frac{10}{1000} = 0.01$ and $P(E|\bar{E}_A) = \frac{2}{2} = 1$,

$$P(E) = 0.01 \cdot \frac{1000}{1000} + 1 \cdot \frac{2}{2} = 0.12$$

Now we can calculate $P(E|\bar{E}_B)$:

$$P(E|\bar{E}_B) = \frac{P(E \cap \bar{E}_B)}{1 - P(E)} = \frac{0.1}{1 - 0.12} = \frac{0.1}{0.88} = \frac{1}{8.8}$$

Therefore, the probability that a randomly selected lemon car came from Factory B is approximately $\frac{1}{8.8}$. 

**Note:** The calculation assumes equal prior probabilities for the two factories, which is not necessarily the case in real-world scenarios. If Factory A produces many more cars than Factory B, the prior probability of a car coming from Factory A would be much higher, and the calculation would need to take this into account.
Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

\[
P(B|L) = \frac{P(L|B)P(B)}{P(L)}
\]

\[
= \frac{2}{1000} \times \frac{1}{1100} = \frac{2}{1100} = \frac{2}{5}
\]
Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$
Total probability:

Sunny 80% to workout
Rainy 30% to workout

# Sunny / # Rainy = 3

\[ P(\text{work out}) = 0.75 \times 0.80 + 0.25 \times 0.30 \]
Total probability:

\[ P(B) = P(A \cap B) + P(A^c \cap B) \]

\[ = P(B|A)P(A) + P(B|A^c) \cdot P(A^c) \]

\[ A \rightarrow \text{sunny} \]

\[ A^c \rightarrow \text{rainy} \]

\[ P(B|A) = \]

\[ P(B|A^c) = \]

\[ P(A) = \]

\[ P(A^c) = \]
Total probability

\[ P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) \]
\[ = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) \]
\[ + P(B \mid A_3)P(A_3) \]

\[ A_1, A_2, A_3 \] are disjoint, \[ A_1 \cup A_2 \cup A_3 = B \]
\[ P(B) = \sum_j P(B \cap A_j) = \sum_j P(B \mid A_j) P(A_j) \]

\( A_j, A_k \) are disjoint
\( A_j \cap A_k = \emptyset \quad j \neq k \)
Total probability:

\[ P(\text{meat}) = ? \]
\[ 0.8 \times 0.5 + 0.2 \times 0.3 \]
\[ = \]

\[ P(\text{soup} \mid \text{meat}) = ? \]
\[ = \frac{P(\text{soup} \cap \text{meat})}{P(\text{meat})} \]
\[ = \frac{0.8 \times 0.5}{0.8 \times 0.5 + 0.2 \times 0.3} \]
<table>
<thead>
<tr>
<th>Soup</th>
<th>Meat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fish</td>
<td>Juice</td>
</tr>
</tbody>
</table>
Bayes rule using total prob.

\[ P(A_j | B) = \frac{P(B | A_j) P(A_j)}{P(B)} \]

\[ = \frac{P(B | A_j) P(A_j)}{\sum_j P(B | A_j) P(A_j)} \]

\[ A_i \cap A_j = \emptyset \quad \text{if} \quad i \neq j \quad \text{disjoint} \]
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \( P(D|T) \), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T)}
= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]
Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is \( \frac{1}{100,000} \). If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is \( P(D|T) \), the probability of having disease given a positive test result?

\[
P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}
\]

\[
P(D|T) < 1.9
\]
Independence

One definition:

\[ P(A|B) = P(A) \text{ or} \]
\[ P(B|A) = P(B) \]

Whether A happened doesn’t change the probability of B and vice versa.
Independence: example

Suppose that we have a fair coin and it is tossed twice. Let $A$ be the event “the first toss is a head” and $B$ the event “the two outcomes are the same.”

$A = \text{HH} \cup \text{TT}$

$P(A|B) = P(A)$

$P(A) = \frac{1}{2}$

$P(A|B) = P(A)$

These two events are independent!
Independence

Alternative definition

\[
P(A|B) = P(A) \\
\frac{P(A \cap B)}{P(B)} = P(A) \\
\Rightarrow P(A \cap B) = P(A)P(B)
\]
Suppose you draw one card from a standard deck of cards. \( E_1 \) is the event that the card is a King, Queen or Jack. \( E_2 \) is the event the card is a Heart. Are \( E_1 \) and \( E_2 \) independent?

\[
P(E_1 \cap E_2) = \frac{3}{52}
\]

\[
P(E_1) = \frac{12}{52} = \frac{3}{13}
\]

\[
P(E_2) = \frac{13}{52} = \frac{1}{4}
\]

Yes
Pairwise independence is not mutual independence in larger context

\[ P(A_1) = P(A_2) = P(A_3) = P(A_4) = \frac{1}{4} \]

\[ A = A_1 \cup A_2; \ P(A) \]

\[ B = A_1 \cup A_3; \ P(B) \]

\[ C = A_1 \cup A_4; \ P(C) \]

\[ P(ABC) \ is \ the \ shorthand \ for \ P(A \cap B \cap C) \]
Mutual independence of a collection of events $A_1, A_2, A_3...A_n$ is:

$$P(A_i | A_j A_k ... A_p) = P(A_i)$$

for $j, k, ... p \neq i$

It’s very strong independence!
An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?

\[ \Pr(A_1 \cap A_2 \cap \cdots \cap A_7) = p^7 \]

[The probability that all 7 people show up.]

\[ = \Pr(A_1) \cdot \Pr(A_2) \cdots \cdot \Pr(A_7) \]
An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that exactly 6 people showed up?

$$P(6 \text{ people showed up}) = \binom{8}{6} p^6 (1 - p)^2$$
An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability $p$, what is the probability that the flight is overbooked?

$$\text{P( overbooked) } = \sum_{u=7}^{8} \binom{8}{u} p^u (1-p)^{8-u}$$
Assignments

❖ Module week3 on Canvas
❖ Next time: Random variable
Additional References

- Charles M. Grinstead and J. Laurie Snell 
  "Introduction to Probability"

- Morris H. Degroot and Mark J. Schervish 
  "Probability and and Statistics"
See you next time

See You!