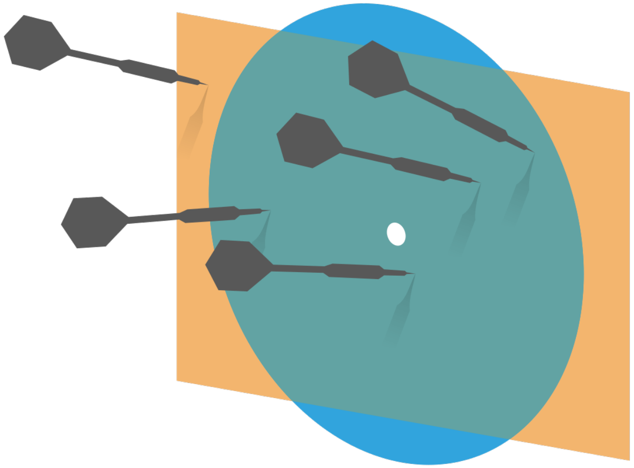
Probability and Statistics for Computer Science



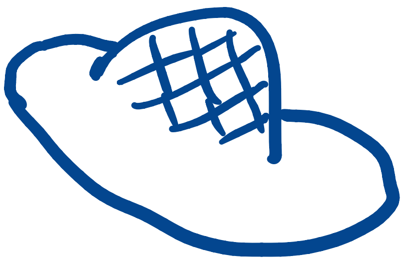
Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.10.2021

Counting: how many ways?

if we put 7 hats (indistinguishable) on 7 people out of 10 people randomly?



Warm up: which is larger?

$P(A \cap B)$ or $P(A | B)$

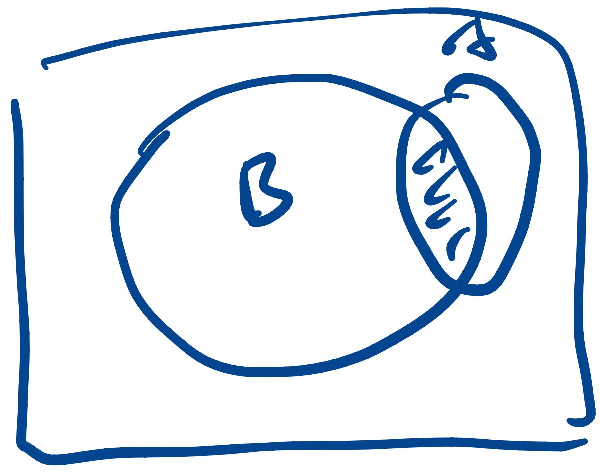
$P(B) < 1$

A) $P(A \cap B)$

$\boxed{\text{B)}}$ $P(A | B)$

C) None

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Last time

* More probability calculation using counting

* Conditional probability

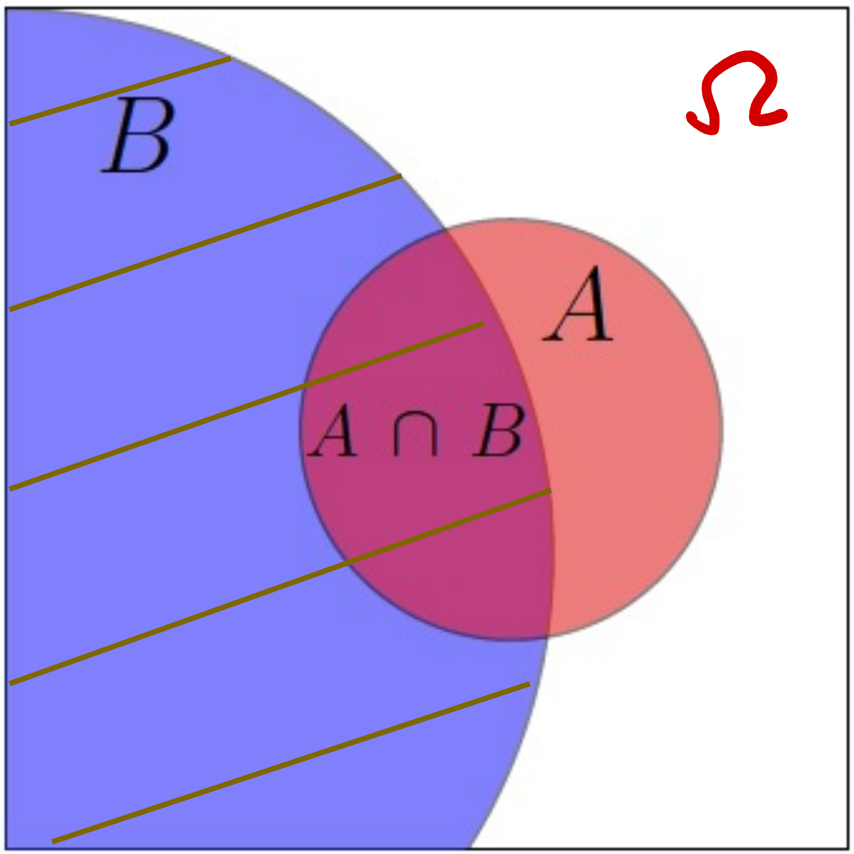
  Mutiplication rule;

Objectives

- Conditional Probability
  - Review
  - Bayes rule
  - Total probability
  - Independence

Conditional Probability

- The probability of ***A*** given ***B***



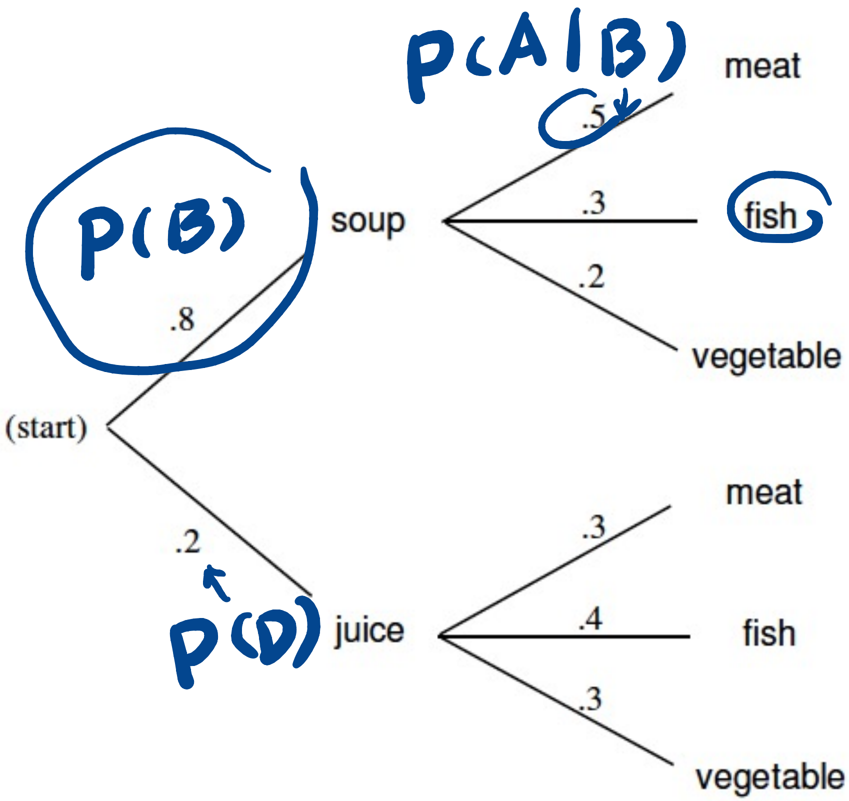
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The line-crossed area is the new sample space for conditional P(A| B)

Joint Probability Calculation

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

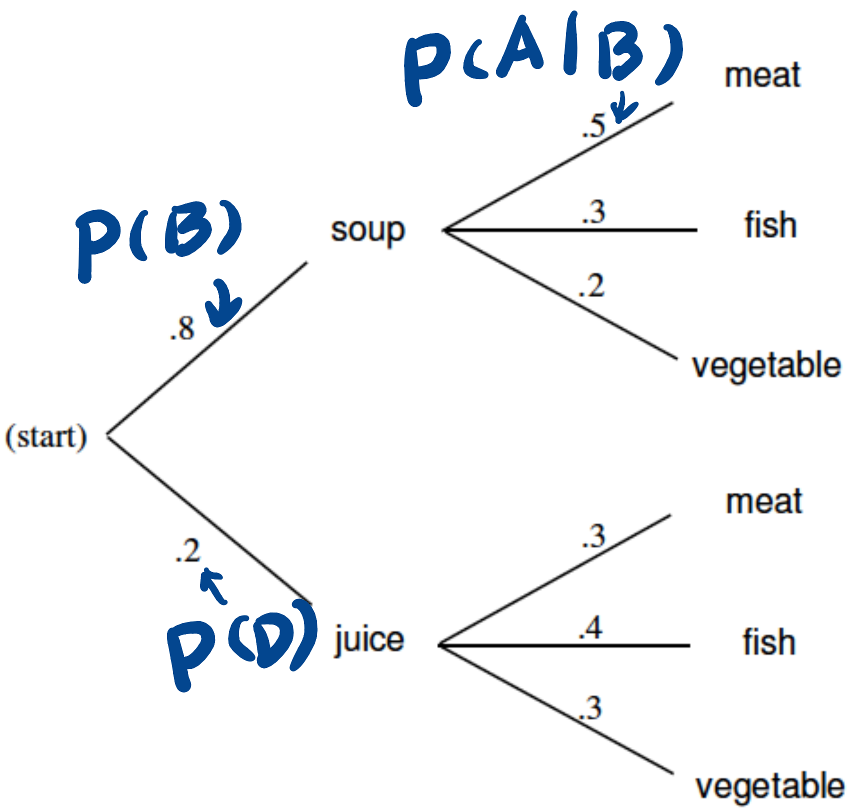


$\rightarrow$ 0.8×0.3

$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Joint Probability Calculation

$$\Rightarrow P(A \cap B) = P(A|B)\underline{P(B)}$$ prior



$P(A^c | B) = ?$ $1 - P(A|B)$

$P(A^c \cap D) = ?$

$1 - P(A \cap D)$

A: meat B: soup

D: juice

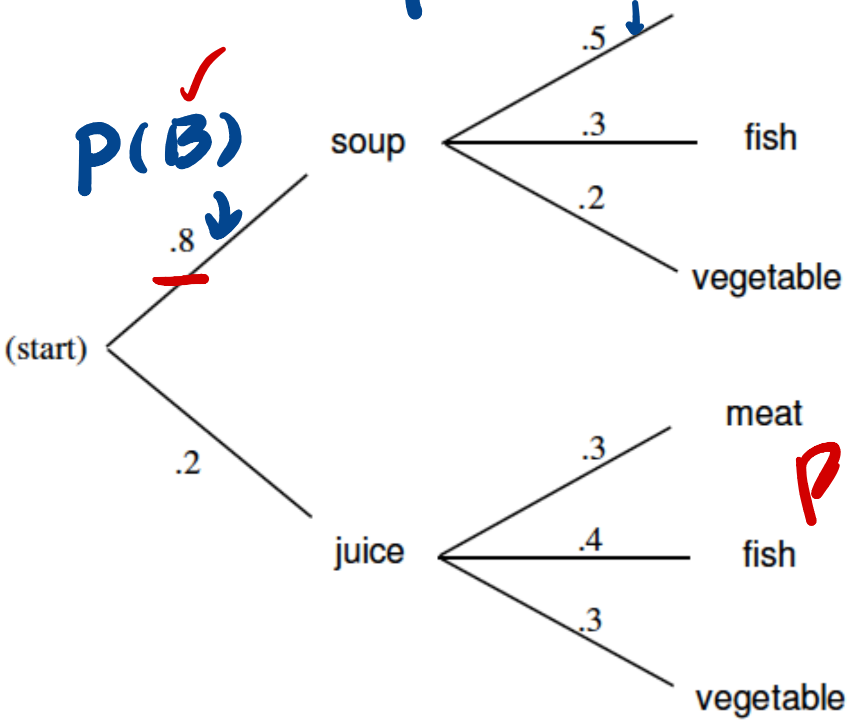
$$P(soup \cap meat) = P(meat|soup)P(soup) = 0.5 \times 0.8 = 0.4$$

Joint Probability Calculation

$$\Rightarrow P(A\cap B) = P(A|B)P(B)$$

$P(A|B)$

$P(B)$



0.6 icecream

$P(C|B\cap A)$ cheesecake ✓

0.4

C: cheesecake

$$P(C|B) = ?$$

$$= \frac{P(B\cap C)}{P(B)}$$

$$P(B\cap C) = P(B\cap A\cap C) + P(B\cap A^c\cap C)$$

$$P(soup\cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5\times 0.8 = 0.4$$

Symmetry of joint event in terms of conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$
$$\Rightarrow P(B \cap A) = P(B|A)P(A)$$

Symmetry of joint event in terms of conditional prob.

$\because \quad B \cap A = A \cap B$

$$P(B \cap A) = P(A \cap B)$$

$$\Rightarrow$$

$$\boxed{P(A|B)P(B) = P(B|A)P(A)}$$

The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

P(A|B) =?

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A|D) = \frac{P(D|A)\,P(A)}{P(D)}$$

The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

$P(A|B) \stackrel{?}{=}$

$$\Longrightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thomas Bayes (1701-1761)

Bayes rule: lemon cars

There are two car factories, **A** and **B**, that supply the same dealer. Factory **A** produced **1000** cars, of which **10** were lemons. Factory **B** produced **2** cars and both were lemons. You bought a car that turned out to be a lemon. E_B

What is the probability that it came from factory **B**? E_A

E_B : lemon

E_A : it came Fac B

$$P(E_A \mid E_B) = ? = \frac{P(E_B \mid E_A) \cdot P(E_A)}{P(E_B)}$$

Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

$$P(B|L) = \frac{P(L|B)P(B)}{P(L)} = \frac{1 \times \frac{2}{1002}}{\frac{12}{1002}} = \frac{2}{12} = \frac{1}{6}$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

Total probability:

Sunny 80% to workout

Rainy 30% to workout

Sunny / # Rainy = 3

P(Rainy ∩ out)

P(Sunn ∩ out) P(work out) = ?

75% × 80% + 25% × 30%

Total probability:

$$P(B) = P(A \cap B) + P(A^c \cap B)$$
$$= P(B|A)P(A) + P(B|A^c) \cdot P(A^c)$$

$A \rightarrow$ sunny

$A^c \rightarrow$ rainy

$P(B|A) =$

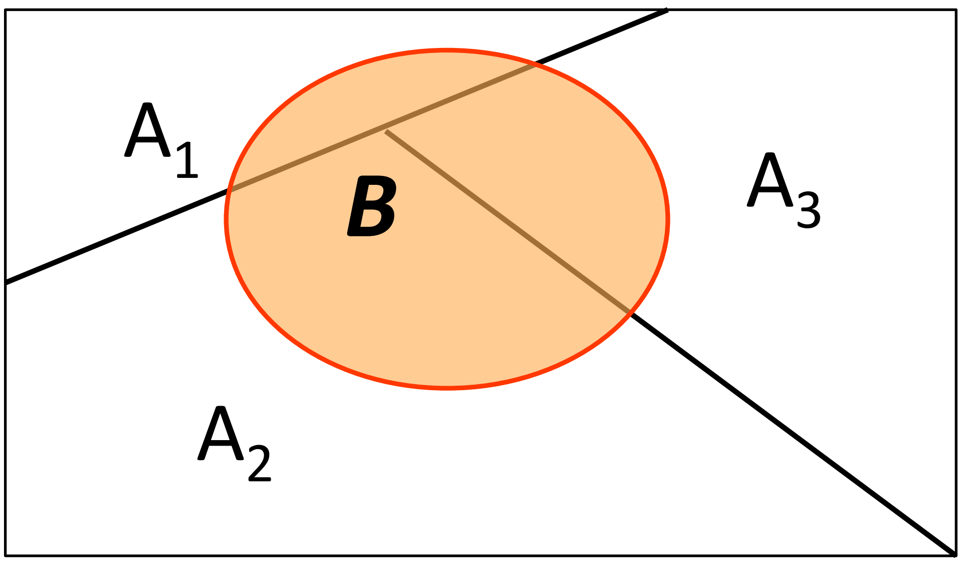
$P(B|A^c) =$

$P(A) =$

$P(A^c) =$

Total probability

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$
$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$
$$+ P(B|A_3)P(A_3)$$



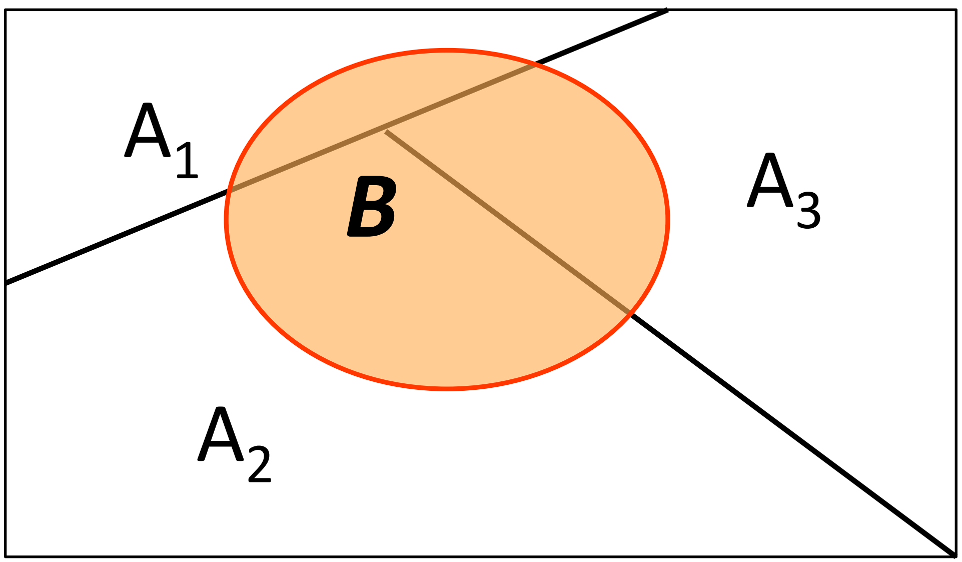
A_1, A_2, A_3 are disjoint,

$A_1 \cup A_2 \cup A_3 \supseteq B$

Total probability general form

$$P(B) = \sum_j P(B \cap A_j)$$

$$= \sum_j P(B \mid A_j)\, P(A_j)$$

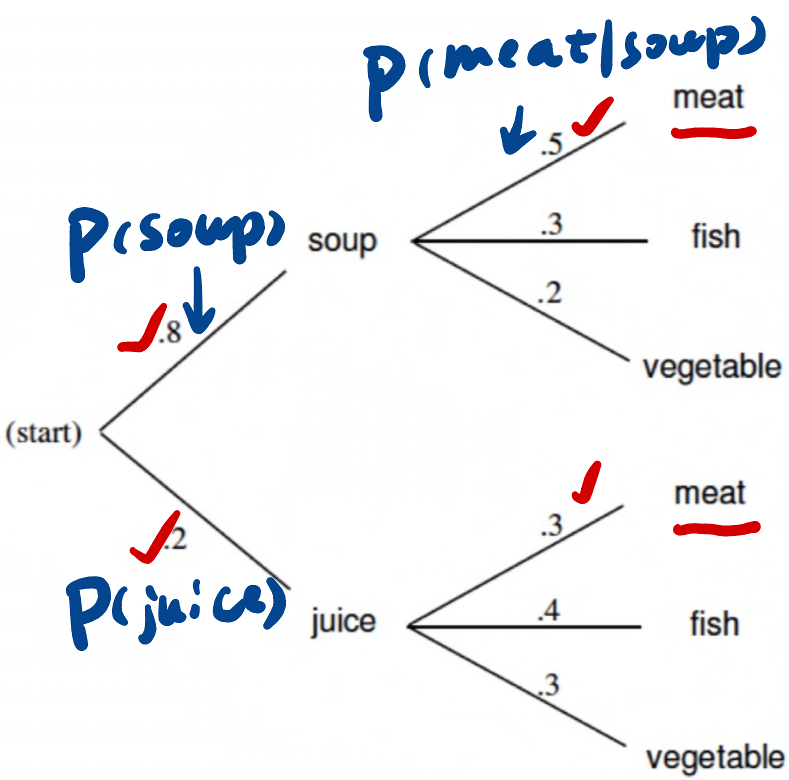


A_j, A_k are disjoint

$A_j \cap A_k = \phi$

$j \neq k$

Total probability:



$P(\text{meat}) = ?$

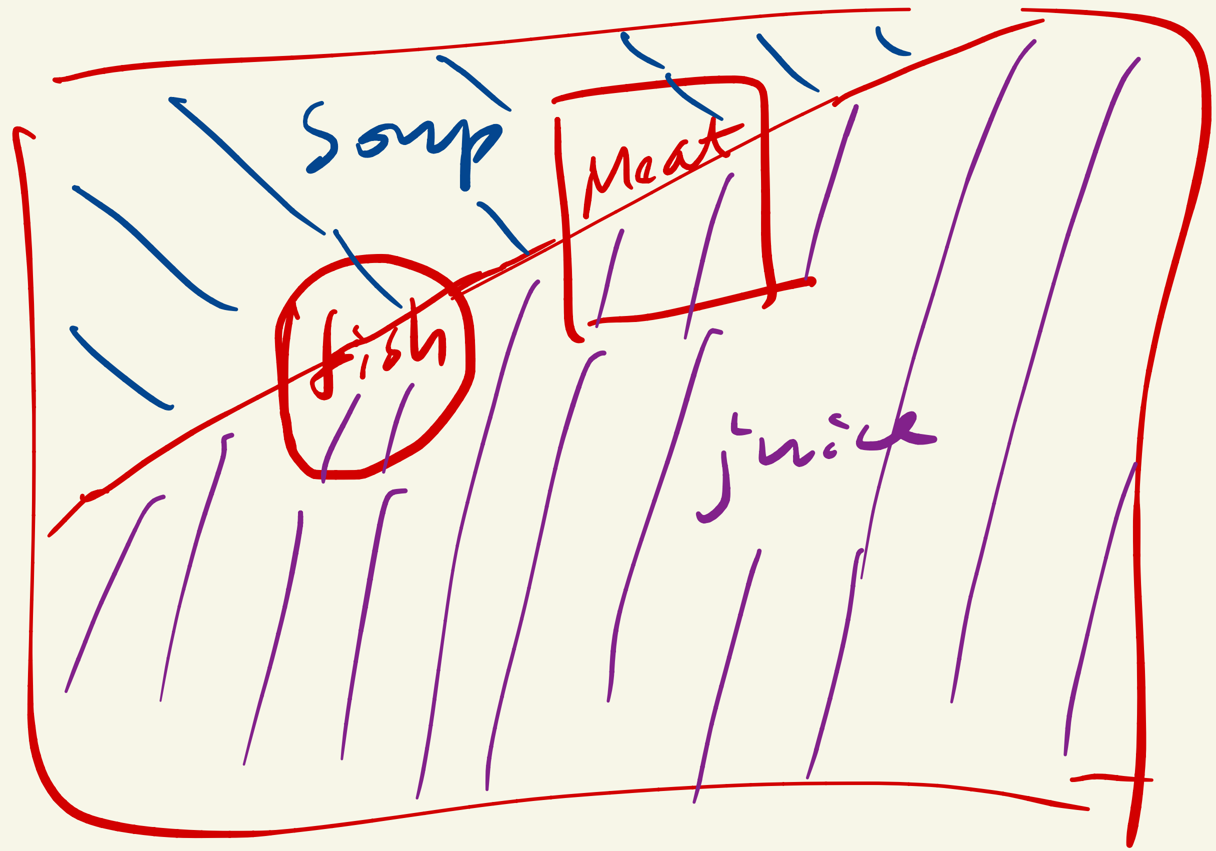
$0.8 \times 0.5 + 0.2 \times 0.3$

$=$

$P(\text{soup} \mid \text{meat}) = ?$

$= \dfrac{P(\text{soup} \cap \text{meat})}{P(\text{meat})}$

$= \dfrac{0.8 \times 0.5}{0.8 \times 0.5 + 0.2 \times 0.3}$


Soup
Meat
fish
juice

Bayes rule using total prob.

$$P(A_j | B) = \frac{P(B|A_j)\,P(A_j)}{P(B)}$$

$$= \frac{P(B|A_j)\,P(A_j)}{\sum_j P(B|A_j)\,P(A_j)}$$

$A_i \cap A_j = \phi$ → disjoint

if $i \neq j$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

Using total prob.

$P(T|D)P(D) + P(T|D^c)P(D^c)$

$$= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is **1/100,000**. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability **0.001**. What is $P(D|T)$, the probability of having disease given a positive test result?

$$0.95 \times \frac{1}{100,000}$$

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)\underline{P(D^c)}}$$

$$0.95 \times \frac{1}{10^5} + 0.001\,(1 - P(D))$$

$$P(T|D^c) = 0.001$$

$$P(D|T) < 1\%$$

Independence

- One definition:

$$P(A|B) = P(A) \; or$$
$$P(B|A) = P(B)$$

Whether A happened doesn't change the probability of B and vice versa

Independence: example

- Suppose that we have a fair coin and it is tossed twice. let A be the event "the first toss is a head" and B the event "the two outcomes are the same."

$A: H*$ $\quad B: HH \quad TT$

$P(A|B) \stackrel{?}{=} P(A)$

$P(A) = \frac{1}{2}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HH\})}{P(\{HH, TT\})} = \frac{1}{2}$

$P(A|B) = P(A)$

- These two events are independent!

Independence

- Alternative definition

$$P(A|B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$$

Testing Independence:

- Suppose you draw one card from a standard deck of cards. E_1 is the event that the card is a King, Queen or Jack. E_2 is the event the card is a Heart. Are E_1 and E_2 independent?

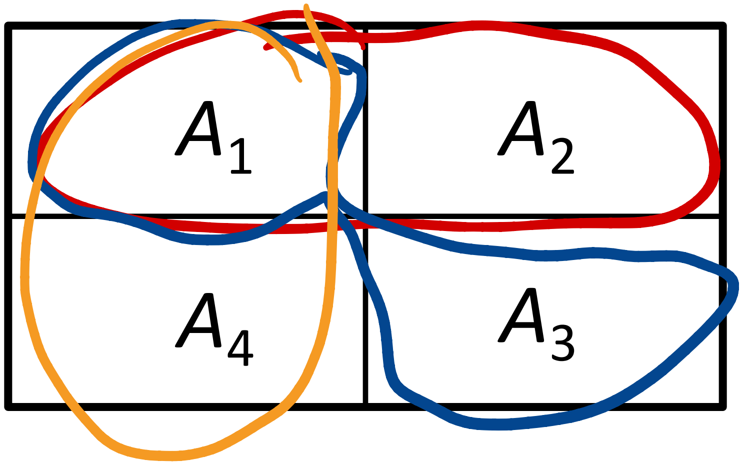
Yes

$$P(E_1 \cap E_2) = \frac{3}{52}$$

$$P(E_1) = \frac{12}{52}$$

$$P(E_2) = \frac{13}{52} = \frac{1}{4}$$

Pairwise independence is not mutual independence in larger context



P(A_1) = P(A_2) = P(A_3) = P(A_4) = 1/4

$P(A \cap B) = P(A_1) = \frac{1}{4} \leftarrow P(A)P(B) = \frac{1}{2} \times \frac{1}{2}$

$P(A \cap C) = P(A_1) = \frac{1}{4}$

$P(B \cap C) = P(A_1) = \frac{1}{4}$

$P(A \cap B \cap C) = P(A_1) = \frac{1}{4}$

$\neq P(A)P(B)P(C)$

$= \frac{1}{8}$

$$A = A_1 \cup A_2; P(A)$$

$$B = A_1 \cup A_3; P(B)$$

$$C = A_1 \cup A_4; P(C)$$

*$P(ABC)$ *is the shorthand for* $P(A \cap B \cap C)$

Mutual independence

- Mutual independence of a collection of events $A_1, A_2, A_3 \ldots A_n$ is :

$$P(A_i|A_jA_k\ldots A_p) = P(A_i)$$

$$j, k, \ldots p \neq i$$

- It's very strong independence!

Probability using the property of Independence: Airline overbooking (1)

- An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

7 showed up

$$P(A_1 \cap A_2 \cap A_3 \cdots \cap A_7) = p^7$$

$$= P(A_1)P(A_2) \cdots P(A_7)$$

Probability using the property of Independence: Airline overbooking (2)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that exactly 6 people showed up?

tickets

1 2 3 4 5 6 _ _

P(6 people showed up) = $\binom{8}{6} p^6 (1-p)^2$

$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7^c \cap A_8^c)$

Probability using the property of Independence: Airline overbooking (3)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P(overbooked) = $\sum_{u=7}^{8} \binom{8}{u} p^u (1-p)^{8-u}$

Assignments

- Module week3 on Canvas
- Next time: Random variable

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

See you next time

See You!