I have now used each of the terms mean, variance, covariance and standard deviation in two slightly different ways.

Forsythe
Last time:
- Probability
- Conditional probability and joint
- Cumulative distribution
- Probability distribution
- Random Variable
Objectives

• Variance & covariance
• Expected value
• Independence of random variables
• Random Variable
Independence of random variables

Random variables $X$ and $Y$ are independent if

\[ P(X = x, Y = y) = P(X = x)P(Y = y) \]

for all $x$ and $y$. In the previous coin toss example, are $S$ and $D$ independent? Are $X$ and $Y$ independent? $P(X = x, Y = y) = P(X = x)P(Y = y)$

Indepedent if random variables $X$ and $Y$ are

\[ P(X = x)P(Y = y) \]

For example, $P(A \cap B) = P(A)P(B)$
Joint Probability Example

Tossing a coin twice, we define random variables $X$ and $Y$ for each toss. For each toss of a coin, we define

$\begin{align*}
X(\omega) &= \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail}
\end{cases} \\
Y(\omega) &= \begin{cases} 
1 & \text{outcome of } \omega \text{ is head} \\
0 & \text{outcome of } \omega \text{ is tail}
\end{cases}
\end{align*}$
Joint probability distribution example

\[
\begin{array}{c|c|c}
X & 0 & 1 \\
\hline
0 & \frac{2}{3} & \frac{1}{3} \\
\hline
1 & \frac{1}{3} & \frac{2}{3} \\
\end{array}
\]

\[P(X=0) = \frac{2}{3}, \quad P(X=1) = \frac{1}{3}\]

\[P(Y=0 | X=0) = \frac{2}{3}, \quad P(Y=0 | X=1) = \frac{1}{3}\]

\[P(Y=1 | X=0) = \frac{1}{3}, \quad P(Y=1 | X=1) = \frac{2}{3}\]
Joint probability distribution

\[ P(s, d) = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \]

\[ P(s) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

\[ P(d) = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

\[ P(s=0, d=0) = 0 \]

\[ P(s=1, d=0) = \frac{1}{2} \]

\[ P(s=1, d=1) = \frac{1}{4} \]

\[ P(s=1, d=2) = \frac{1}{4} \]

\[ \text{Example} \]

Joint probability distribution

\[ P(s', p) = \begin{bmatrix} 
\begin{array}{ccc}
0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} \\
0 & \frac{1}{4} & 0 \\
\end{array}
\end{bmatrix} \]
Joint probability distribution

\[(0 = D)p(1 = S) \neq (0 = D, 1 = S)p\]

Example of probability distribution
Bayes rule for random variables generalizes to events:

\[
\frac{(x)d(x|h)d_x}{(x)d(x|h)d} = \frac{(h)d}{(x)d(x|h)d} = (h|x)d
\]

Total Probability:

\[
\frac{(B)d}{(V)d(V|B)d} = (B|V)d
\]
After class

Conditional probability distribution

Example

\[ \frac{(p)_d}{(p', s)_d} = (p | s)_d \]
\[
\frac{\frac{\mathbb{I}}{D}}{\frac{\mathbb{I}}{I} \times \mathbb{I}} = \frac{(I = S)D}{(I = A)D(I = A|I = S)D} = (I = S|I = A)D
\]

\[
\begin{array}{ccc}
0 & \frac{\mathbb{I}}{I} & 0 \\
1 & 0 & 1 \\
0 & \frac{\mathbb{I}}{I} & 0
\end{array}
\]

\[
\frac{(p)D}{(p,s)D} = (p|s)D
\]
Three important facts of random variables:

- Random variables have probability functions.
- Random variables can be conditioned on events or other random variables.
- Random variables have averages.

Random variables can be probability functions.
The expected value of a random variable $X$ can take all the values that $X$ can take. The expected value is a weighted sum:

$$E[X] = \sum_{x} x \cdot P(x)$$

where $P(x)$ is the probability of the random variable $X$ taking the value $x$. This represents the expected value (or expectation) of $X$. The expected value is a measure of the long-run average of the random variable $X$.
The expected value of a random variable $X$ is a weighted sum of all the values $X$ can take. The expected value is given by:

$$E[X] = \sum x P(x) \leq 1$$

Theoretical mean $p = 0.5$ if $E[X] = 0.5$. The expected value of a random variable $X$ is:

$$E[X] = \sum x P(x)$$
A company has a project that has probability of earning 10 million and probability of losing 10 million.

Let $X$ be the return of the project.

What is $E[X]$?

Let $X$ be the return of the project. What is $E[X]$?

$E[X] = \sum x p(x) = 10p + (-10)(1-p)$

$E[X] = \{ x, d_x \}$

After class

Cookie
Chocolate

$2 each

$1 each

Expected value?

When the two are the same, you get the prize.

Expected value?

(3) Random draw twice with replacement

Out of 4

Random draw 1

Random draw 1

After class
For random variables $X$ and $Y$ and constants $k, c$

- **Additivity**
  \[ E[X + Y] = E[X] + E[Y] \]

- **Scaling property**
  \[ E[kX] = kE[X] \]

And $E[kX + c] = kE[X] + c$
Proof of the additive property

\[ E[X + Y] = E[X] + E[Y] \]
X, Y, Z are Bernoulli with \( p = 0.5 \)

\[
E[X + 2T - Z] = E[X] + 2E[T] - E[Z] = \sqrt{0.5} + 2 \times 0.5 + 0.5 = 1
\]

\[
E[1 \times Z] = E[T] = E[1 \times 2] = \sqrt{0.5} \times 1 = 0.5
\]

\[
= p \times (1 + (1-p).0) = p = 0.5
\]
\[ E \{ x \} + \mathbb{E} \{ Y \} = \mathbb{E} \{ x \} + \mathbb{E} \{ x \} \]

\[ (h, p(x)) + (h, p(x)) = (h, p(x)) + (h, p(x)) \]

\[ h \times p(x) + h \times p(x) = h \times p(x) + h \times p(x) \]

\[ i = 1, 2, \ldots \]

Proof cont.:
Q. What’s the value?

What is $E[E[X]+1]$?

A. $E[X]+1$

B. 1

C. 0
If $f$ is a function of a random variable $X$, then $Y = f(X)$ is a random variable too.

The expected value of $Y = f(X)$ is

$$E[Y] = \sum_y y \cdot P(Y = y)$$

$$E[f(X)] = \sum_x f(x) \cdot P(X = x)$$

$$E[f(X)] = \int_{-\infty}^{\infty} f(x) \cdot f_X(x) \, dx$$
If $f$ is a function of a random variable $X$, then $Y = f(X)$ is a random variable too.

The expected value of $Y$ is

$$E[Y] = E[f(X)] = \sum_x f(x) P(x)$$
The exchange of variable theorem

\[ E[f(x)p(x)] = \int f(x) p(x) \, dx = \int \{ \text{if } \{ x \in \mathbb{X} \} \text{ is bijection } p(x) = p(x) \} \]

\[ E[f(x)] = \int f(x) \, dx = \int \text{if } \{ x \in \mathbb{X} \} \text{ is bijection } p(x) = p(x) \]
A cat moves with random constant speed \( V \), either 5 mile/hr or 20 mile/hr with equal probability, what's the expected time for it to travel 50 miles?

\[
\frac{\text{expected time}}{50} = \frac{2}{5} \times \frac{50}{50} + \frac{2}{1} \times \frac{5}{50}
\]

\[
\text{expected time} = \frac{5}{50}
\]
Q: Is this statement true?

If there exists a constant such that \( P(X \geq a) = 1 \), then \( \mathbb{E}[X] \geq a \).

A. True
B. False

\( p(X \geq a) = 1 \), then if there exists a constant such that

after class
Variance and standard deviation

The variance of a random variable \( X \) is

\[
\text{Var}[X] = \text{std}[X]^2
\]

The standard deviation of a random variable \( X \) is

\[
\text{std}[X] = \sqrt{\text{var}[X]}
\]

The variance of a random variable \( X \) is

\[
\text{Var}[X] = \mathbb{E} \left[ (X - \mathbb{E}[X])^2 \right]
\]
For random variable $X$ and constant $k$:

\[ \text{var}[kX] = k^2 \text{var}[X] \]

\[ 0 \leq \text{var}[X] \]

Properties of Variance

For random variable $X$ and constant $k$.
A neater expression for variance

Variance of Random Variable $X$ is defined as:

$$\text{var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

It's the same as:

$$\mathbb{E}[(X - \mathbb{E}[X])^2] = \text{var}(X)$$
A neater expression for variance:

$$\text{var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
Variance of the profit example

For the profit example, what is the variance of the return? We know

\[
\text{Var} = \mathbb{E}[X^2] - \left(\mathbb{E}[X]\right)^2
\]

\[
= (10p^2 + (1-p)) - \left(20p - 10\right)^2
\]

\[
= 400p(1-p)
\]

variance of the return? We know E[X] = [X] = [X]

For the profit example, what is the variance of the profit example? 

\[
\text{Var} = 400p(1-p)
\]
Motivation for covariance

Study the relationship between random variables.

Applications include the fire control of radar, communicating in the presence of noise.

Note that it's the un-normalized correlation.
The covariance of random variables $X$ and $Y$ is

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

Note that

Let $X = Y$

$$\text{Cov}(X, X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \text{Var}(X).$$

The covariance of random variables $X$ and $Y$ is

Covariance
A neater form for covariance

\[ \text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y] \]

for variance

covariance (similar derivation as

\[ \text{var}(X) = \text{cov}(X, X) = \mathbb{E}[X^2] - \mathbb{E}[X] \mathbb{E}[X] \]

A neater expression for

\[ \mathbb{E}[X] \mathbb{E}[Y] - \mathbb{E}[XY] = \mathbb{E}[X, Y] \]

A neater form for covariance
**Correlation coefficient is normalized covariance**

The correlation coefficient is

\[
\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}
\]

When \(X, Y\) takes on values with equal probability to generate data sets \(\{(x, y)\}\), the correlation coefficient will be as seen in Chapter 2.
Correlation coefficient is normalized covariance

The correlation coefficient can also be written as:

\[ \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} \]

\[ \text{corr}(X, Y) = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \]
Correlation seen from scatter plots

- Positive correlation
- Negative correlation
- Zero correlation

Credit: Prof. Forsyth
Covariance seen from scatter plots

Positive Covariance

Zero Covariance

Negative Covariance

Credit: Prof. Forsyth
When correlation coefficient or covariance is zero:

- The covariance is 0.

That is:

$$E[XY] = E[X]E[Y]$$

and

$$0 = E[XY] - E[X]E[Y] = \text{Cov}(X,Y)$$

which implies

$$\text{Cov}(X,Y) = 0$$

This is a necessary property of independence of random variables, not equal to independence.
Variance of the sum of two random variables

\[ \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] + 2 \text{cov}(X, Y) \]
If events X and Y are independent, then

\[ \mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y] \]

If \( X \) and \( Y \) are independent, \( \mathbb{P}(X, Y) = \mathbb{P}(X) \mathbb{P}(Y) \)

\[ \mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y] \]
Assignments

Finish week 4 module

Next time: Markov and Chebyshev inequality & Weak law of Large numbers, Continuous random variable

Finish week 4 module at 3:30pm

Quiz 3 at 4:30pm
Charles M. Grinstead and J. Laurie Snell
"Introduction to Probability"

Morris H. Degroot and Mark J. Schervish
"Probability and Statistics"
See you next time!

See you!

[Image of a charm bracelet with a pink elephant and a green four-leaf clover charm]