“I have now used each of the terms mean, variance, covariance and standard deviation in two slightly different ways.” ---Prof. Forsythe
Last time

- Random Variable
- Probability distribution
- Cumulative distribution
Objectives

- Random Variable
  - Independence of random variables
  - Expected value
  - Variance & covariance
Independence of random variables

Random variable $X$ and $Y$ are independent if

$$P(x, y) = P(x)P(y) \text{ for all } x \text{ and } y$$

In the previous coin toss example

Are $X$ and $Y$ independent?
Are $S$ and $D$ independent?
### Joint probability distribution example

**$P(x, y)$**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

**$X$**

<table>
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<th>0</th>
<th>1</th>
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<tbody>
<tr>
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<td>$\frac{1}{2}$</td>
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<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
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</tbody>
</table>

**$P(y)$**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
Joint probability distribution example

\[ P(s, d) \]

<table>
<thead>
<tr>
<th>( s )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( \frac{1}{4} )</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ P(s) \]

- \( \frac{1}{4} \)
- \( \frac{1}{2} \)
- \( \frac{1}{4} \)

\[ P(d) \]

- \( \frac{1}{4} \)
- \( \frac{1}{2} \)
- \( \frac{1}{4} \)
Joint probability distribution example

\[ P(s, d) \]

\[
\begin{array}{ccc}
0 & 1 & 0 \\
\frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & 0 \\
\end{array}
\]

\[ P(s) \]

\[
\begin{array}{c}
\frac{1}{4} \\
\frac{1}{2} \\
\frac{1}{4} \\
\end{array}
\]

\[ P(d) \]

\[
\begin{array}{c}
\frac{1}{4} \\
\frac{1}{2} \\
\frac{1}{4} \\
\end{array}
\]

\[ P(S = 1, D = 0) \neq P(S = 1)P(D = 0) \]
Conditional probability distribution example

\[ P(s|d) = \frac{P(s, d)}{P(d)} \]

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
\hline
0 & 0 & \frac{1}{2} & 0 \\
1 & 1 & 0 & 1 \\
2 & 0 & \frac{1}{2} & 0 \\
\end{array}
\]
Bayes rule for random variable

Bayes rule for events generalizes to random variables

\[
P(A|B) = \frac{P(B|A)P(A)}{P(B)}
\]

\[
P(x|y) = \frac{P(y|x)P(x)}{P(y)}
= \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)}
\]

Total Probability
### Conditional probability distribution example

\[ P(s \mid d) = \frac{P(s, d)}{P(d)} \]

\[
\begin{array}{c|ccc}
S & -1 & 0 & 1 \\
\hline
0 & 0 & \frac{1}{2} & 0 \\
1 & 1 & 0 & 1 \\
2 & 0 & \frac{1}{2} & 0 \\
\end{array}
\]

\[
P(D = -1 \mid S = 1) = \frac{P(S = 1 \mid D = -1)P(D = -1)}{P(S = 1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}}
\]
Three important facts of Random variables

- Random variables have probability functions
- Random variables can be conditioned on events or other random variables
- Random variables have averages
The expected value (or expectation) of a random variable $X$ is

$$E[X] = \sum_{x} xP(x)$$

The expected value is a weighted sum of all the values $X$ can take.
The expected value of a random variable $X$ is

$$E[X] = \sum_{x} xP(x)$$

The expected value is a weighted sum of all the values $X$ can take.

$\leq 1$
Expected value: profit

- A company has a project that has \( p \) probability of earning 10 million and \( 1-p \) probability of losing 10 million.

- Let \( X \) be the return of the project.
For random variables $X$ and $Y$ and constants $k, c$

Scaling property

$$E[kX] = kE[X]$$

Additivity

$$E[X + Y] = E[X] + E[Y]$$

And

$$E[kX + c] = kE[X] + c$$
Linearity of Expectation

Proof of the additive property

\[ E[X + Y] = E[X] + E[Y] \]
Q. What’s the value?

What is $E[E[X]+1]$?

A. $E[X]+1$    B. 1    C. 0
If $f$ is a function of a random variable $X$, then $Y = f(X)$ is a random variable too.

The expected value of $Y = f(X)$ is
If \( f \) is a function of a random variable \( X \), then \( Y = f(X) \) is a random variable too.

The expected value of \( Y = f(X) \) is

\[
E[Y] = E[f(X)] = \sum_x f(x)P(x)
\]
Expected time of cat

A cat moves with random constant speed \( V \), either 5mile/hr or 20mile/hr with equal probability, what’s the expected time for it to travel 50 miles?
Q: Is this statement true?

If there exists a constant such that \( P(X \geq a) = 1 \), then \( E[X] \geq a \). It is:

A. True
B. False
The variance of a random variable $X$ is

$$\text{var}[X] = E[(X - E[X])^2]$$

The standard deviation of a random variable $X$ is

$$\text{std}[X] = \sqrt{\text{var}[X]}$$
Properties of variance

- For random variable $X$ and constant $k$

\[ \text{var}[kX] = k^2 \text{var}[X] \]

\[ \text{var}[X] \geq 0 \]
A neater expression for variance

Variance of Random Variable X is defined as:

\[ \text{var}[X] = E[(X - E[X])^2] \]

It’s the same as:

\[ \text{var}[X] = E[X^2] - E[X]^2 \]
A neater expression for variance

\[ \text{var}[X] = E[(X - E[X])^2] \]
A neater expression for variance

\[ \text{var}[X] = E[(X - E[X])^2] \]

\[ \text{var}[X] = E[(X - \mu)^2] \quad \text{where} \quad \mu = E[X] \]
A neater expression for variance

\[
\text{var}[X] = E[(X - E[X])^2]
\]

\[
\text{var}[X] = E[(X - \mu)^2] \quad \text{where } \mu = E[X]
\]

\[
= E[X^2 - 2X\mu + \mu^2]
\]
Variance: the profit example

For the profit example, what is the variance of the return? We know $E[X] = 20p - 10$

$$\text{var}[X] = E[X^2] - (E[X])^2$$
Motivation for covariance

- Study the relationship between random variables
- Note that it’s the un-normalized correlation
- Applications include the fire control of radar, communicating in the presence of noise.
Covariance

- The covariance of random variables $X$ and $Y$ is

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

- Note that

$$cov(X, X) = E[(X - E[X])^2] = var[X]$$
A neater form for covariance

A neater expression for covariance (similar derivation as for variance)

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$
The correlation coefficient is normalized covariance

\[ \text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y} \]

When \( X, Y \) takes on values with equal probability to generate data sets \( \{(x,y)\} \), the correlation coefficient will be as seen in Chapter 2.
The correlation coefficient can also be written as:

\[
\text{corr}(X, Y) = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y}
\]
Correlation seen from scatter plots

Zero Correlation

Positive correlation

Negative correlation

Credit: Prof. Forsyth
Covariance seen from scatter plots

Zero Covariance
No Correlation

Positive Covariance
Positive Correlation

Negative Covariance
Negative Correlation

Credit: Prof. Forsyth
When correlation coefficient or covariance is zero

- The covariance is 0!
- That is:

\[ E[XY] - E[X]E[Y] = 0 \]

\[ E[XY] = E[X]E[Y] \]

- This is a necessary property of independence of random variables * (not equal to independence)
Variance of the sum of two random variables

\[ \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] + 2\text{cov}(X, Y) \]
If events X & Y are independent, then

\[ E[XY] = E[X]E[Y] \]
Proof:

\[ E[XY] = E[X]E[Y] \]
These are equivalent! Uncorrelatedness

\[ E[XY] = E[X]E[Y] \]

\[ \text{cov}(X, Y) = 0 \]

\[ \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \]
We toss two identical coins A & B independently for three times and 4 times respectively, for each head we earn $1, we define $X$ is the earning from A and $Y$ is the earning from B. What is $E(XY)$?

A. $2  
B. $3  
C. $4
If two random variables are uncorrelated, does this mean they are independent? Investigate the case $X$ takes -1, 0, 1 with equal probability and $Y=X^2$. 
Covariance example

It’s an underlying concept in principal component analysis in Chapter 10
Assignments

Finish week4 module

Next time: Markov and Chebyshev inequality & Weak law of large numbers, Continuous random variable
Additional References

• Charles M. Grinstead and J. Laurie Snell
  "Introduction to Probability"

• Morris H. Degroot and Mark J. Schervish
  "Probability and Statistics"
See you next time

See You!