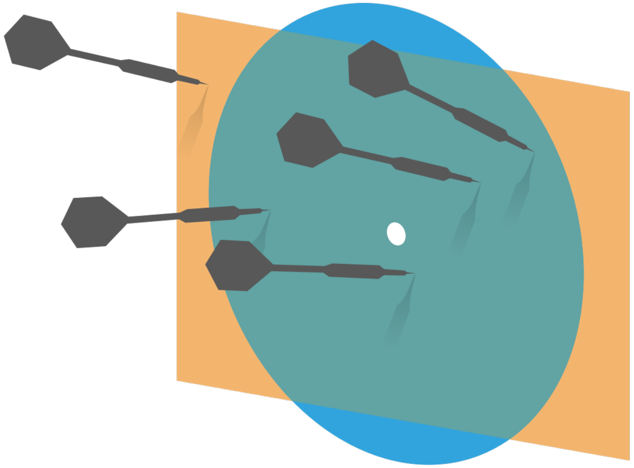
# Probability and Statistics for Computer Science



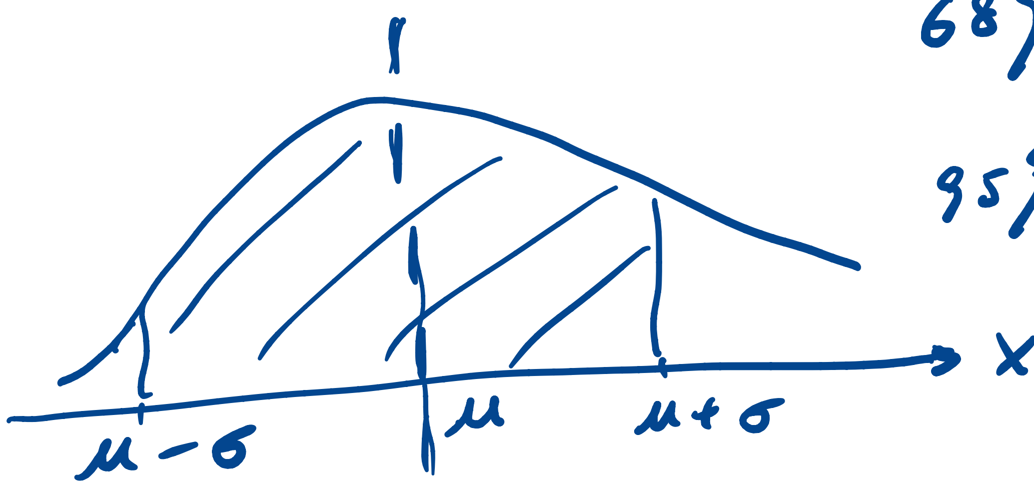
Credit: wikipedia

"In statistics we apply probability to draw conclusions from data."
---Prof. J. Orloff

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 3.09.2021

# Last time

- Exponential distribution
- Normal (Gaussian) distribution




68% $\mu \pm \sigma$

95% $\mu \pm 2\sigma$

# Objectives

- Sample mean
- confidence interval
- t-distribution

# Motivation for drawing conclusion from samples

- In a study of new-born babies’ health, random samples from different time, places and different groups of people will be collected to see how the overall health of the babies is like.

# Motivation of sampling: the poll example

| | | DATES | POLLSTER | SAMPLE | RESULT | | | | NET RESULT |
|---|---|---|---|---|---|---|---|---|---|
| U.S. Senate | Miss. | NOV 25, 2018 | C+ Change Research | 1,211 LV | Espy | 46% | 51% | Hyde-Smith | Hyde-Smith +5 |

**Source: FiveThirtyEight.com**

- This senate election poll tells us:
  - The sample has 1211 likely voters
  - Ms. Hyde-Smith has realized sample mean equal to 51%
- What is the estimate of the percentage of votes for Hyde-smith?
- How confident is that estimate?

# Population

- What is a population?
  - It's the entire possible data set $\{X\}$
  - It has a countable size $N_p$
  - The population mean $popmean(\{X\})$ is a number
  - The population standard deviation is $popsd(\{X\})$ and is also a number
- The population mean and standard deviation are the same as defined previously in chapter 1

# Population

$\boxed{\{X\}} = \{1, 2, 3, \cdots 12\}$ $N_p = 12$

$\text{popmean}(\{X\}) = ?$ $6.5$

$\text{popstd}(\{X\}) = ?$

$$\sqrt{\frac{\sum(x_i - 6.5)^2}{12}}$$

$3.4\ldots$

# Sample

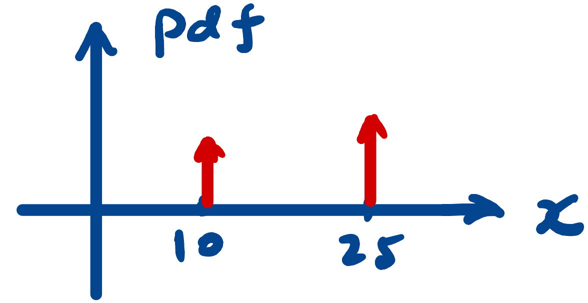
- The sample is a random subset of the population and is denoted as $\{x\}$, where sampling is done with **replacement**
  - The sample size $N$ is assumed to be much less than population size $N_p$
  - The **sample mean of a population** is $X^{(N)}$ and is a **random variable**

# Sample mean example

Money box

10¢ 25¢

dime quarter




* shake and take one and put back.

$X_1$ takes $x_1 = 10$

$E[X] = ?$

$X_2$ takes $x_2 = 10$

$X_3$ takes $x_3 = 25$

$\vdots$

$X_N$ takes $x_N = 10$

$$\boxed{\bar{X}} = \frac{\sum X_i}{N}$$

$$\bar{X} = X^{(N)}$$

# Sample $\{x\}$ and Sample Mean $X^{(N)}$

$N < N_p$

$\{X\} = \{1, 2, 3, - - - \cdot 12\}$

One random sample $\rightarrow$ $\boxed{\{x\}} = \{1, 1, 2, 3, 3\}$ $N=5$

$\frac{10}{5} = 2$

$\boxed{X^{(N)}}$ RV takes a value?

$$X^{(N)} = \frac{x_1 + x_2 + \cdots + x_N}{N}$$

Another random sample $\rightarrow \{1, 1, 1, 1, 1\} \Rightarrow X^{(N)} = 1$

# Sample mean of a population

- The sample mean is the average of **IID** samples

$$\underline{X^{(N)}} = \frac{1}{N}(x_1 + x_2 + ... + x_N) = \text{mean}(\{x\})$$

- By linearity of the expectation and the fact the sample items are identically drawn from the same population with replacement

$$E[X^{(N)}] = \frac{1}{N}(E[X^{(1)}] + E[X^{(1)}].. + E[X^{(1)}]) = E[X^{(1)}]$$

# Expected value of one random sample is the population mean

- Since each sample is drawn uniformly from the population

$X^{(1)}$ $\frac{1}{N_p}$

$$\underline{E[X^{(1)}]} = popmean(\{X\})$$

$= \sum x_i \frac{1}{N_p}$ $= popmean$

therefore $$E[X^{(N)}] = popmean(\{X\})$$

- We say that $X^{(N)}$ is an unbiased estimator of the population mean.

# Standard deviation of the sample mean

- We can also rewrite another result from the lecture on the weak law of large numbers

$$var[X^{(N)}] = \frac{popvar(\{X\})}{N}$$

var[Σ RVᵢ) = Σ var(RVᵢ) if RVᵢ are indpt.

- The standard deviation of the sample mean

$$std[X^{(N)}] = \frac{popsd(\{X\})}{\sqrt{N}}$$

$var(\frac{1}{N} \cdot \Sigma x_i)$

$(\frac{1}{N})^2 var(\Sigma X_i)$

- But we need the population standard deviation in order to calculate the $std[X^{(N)}]$ !

$\frac{1}{N^2} \cdot N \cdot var(x_i)$

$\frac{1}{N} \cdot var(x_i)$

# Unbiased estimate of population standard deviation & Stderr

- The unbiased estimate of $popsd(\{X\})$ is defined as

std({x}) -> popsd

$$stdunbiased(\{x\}) = \sqrt{\frac{1}{N-1} \sum_{x_i \in \ sample} (x_i - mean(\{x_i\}))^2}$$

- So the **standard error** is an estimate of

$$std[X^{(N)}]$$

$$std[X^{(N)}] = \frac{popsd(\{X\})}{\sqrt{N}}$$

approximation

a approxi. of popsd

$$\frac{popsd(\{X\})}{\sqrt{N}} \doteq \frac{stdunbiased(\{x\})}{\sqrt{N}} = \boxed{stderr(\{x\})}$$

# Standard error: election poll

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51%

- What is the estimate of the percentage of votes for Hyde-smith? 51%

$$X_i = \begin{cases} 1 & \text{for Smith} \\ 0 & \text{ow} \end{cases}$$

Number of sampled voters who selected Ms. Smith is:

**1211(0.51) ≅ 618**

$\sum X_i =$ # of votes for Smith

Number of sampled voters who didn't selected Ms. Smith was

**1211(0.49) ≅ 593**

$$\frac{\sum^{N} x_i}{N} \qquad N = 1211$$

# Standard error: election poll

1 1 0 0 . . . . 1

$N = 1211$

- $stdunbiased(\{x\})$

$$= \sqrt{\frac{1}{1211-1}(618(1-0.51)^2 + 593(0-0.51)^2)} = 0.5001001$$

$$X^{(N)} = \frac{x_1 + \dots + x_N}{N}$$

618 "1" for Smith

- $stderr(\{x\})$

$$= \frac{0.5001001}{\sqrt{1211}} \simeq 0.0144$$

593 "0" not for him

# Interpreting the standard error

- **Sample mean** is a random variable and has its own probability distribution, stderr is an estimate of sample mean's standard deviation $X^{(N)} \rightarrow$ Normal if $N \rightarrow \infty$
- When ***N*** is very large, according to the **Central Limit Theorem**, sample mean is approaching a normal distribution with $\mu$ $\sigma$

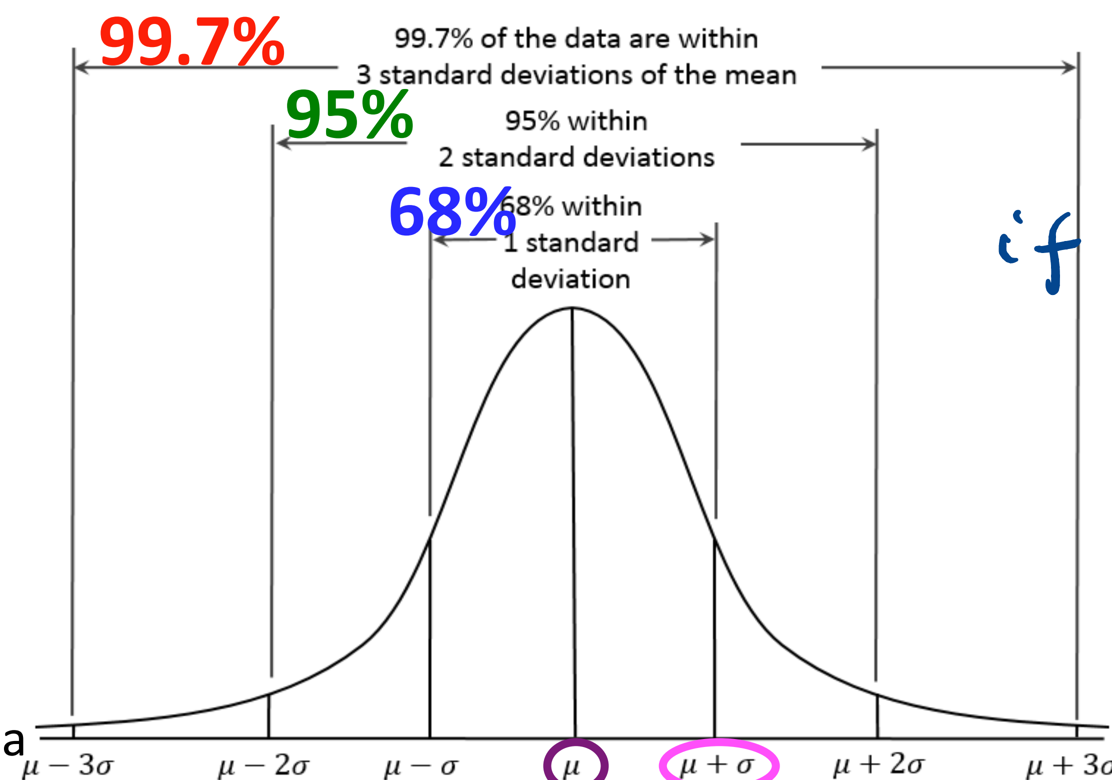
$$\boxed{\mu = popmean(\{X\}) \;\; ; \;\; \sigma = \frac{popsd(\{X\})}{\sqrt{N}} \doteq stderr(\{x\})}$$

$$\boxed{stderr(\{x\}) = \frac{stdunbiased(\{x\})}{\sqrt{N}}}$$

pdf: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\frac{x-\mu}{\sigma})^2/2}$

# Interpreting the standard error

Probability distribution of sample mean tends **normal when N is large**




Credit: wikipedia

if N sample size is large

$\to X^{(N)}$

$\doteq$ mean({x})

Population mean

$\sigma \doteq$ stderr({x})

# Confidence intervals

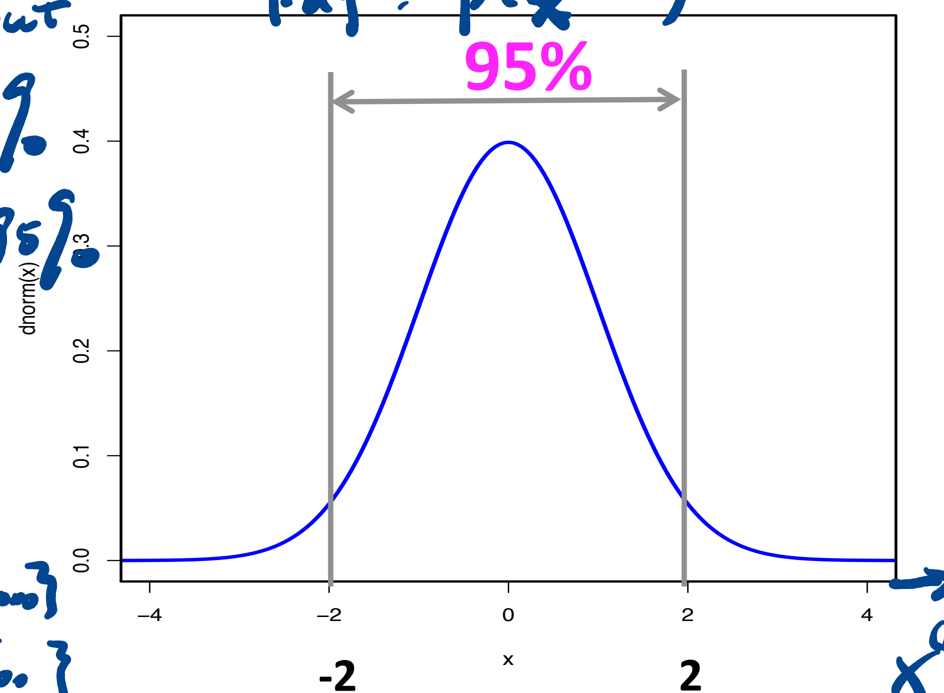
- Confidence interval for a population mean is defined by fraction

input $c\%$

$c\% = 95\%$

- Given a percentage, find how many units of strerr it covers.

pdf : $p(\hat{x}^{(N)})$




$N = 1000$ $\{x_1 \cdots\cdots x_{1000}\}$

$N = 1000$ $\{x_1' \cdots\cdots x_{1000}'\}$

$X^{(N)}$

For **95%** of the **realized sample means**,
the population mean lies in
[sample mean-2 stderr, sample mean+2 stderr]

$std(X^{(N)}) \doteq stderr$

# Confidence intervals when N is large

- For about 68% of realized sample means

$$mean(\{x\}) - stderr(\{x\}) \leq popmean(\{X\}) \leq mean(\{x\}) + stderr(\{x\})$$

- For about 95% of realized sample means

$$mean(\{x\}) - 2stderr(\{x\}) \leq popmean(\{X\}) \leq mean(\{x\}) + 2stderr(\{x\})$$

- For about 99.7% of realized sample means

$$mean(\{x\}) - 3stderr(\{x\}) \leq popmean(\{X\}) \leq mean(\{x\}) + 3stderr(\{x\})$$

# Q. Confidence intervals

- What is the 68% confidence interval for a population mean?

A. [sample mean-2stderr, sample mean+2stderr]
B. [sample mean-stderr, sample mean+stderr]
C. [sample mean-std, sample mean+std]

# Standard error: election poll

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51%

- We estimate the population mean as 51% with stderr 1.44%

- The 95% confidence interval is [51%-2×1.44%,  51%+2×1.44%]= [48.12%, 53.88%]

at your own time

# Q.

- A store staff mixed their fuji  and gala  apples and they were individually wrapped, so they are indistinguishable. if I pick 30 apples and found 21 fuji , what is my 95% confidence interval to estimate the popmean is 70% for fuji? (hint: strerr > 0.05)

A. [0.7-0.17, 0.7+0.17]

B. [0.7-0.056, 0.7+0.056]

# What if N is small? When is N large enough?

- If samples are taken from normal distributed population, the following variable is a random variable whose distribution is Student's **t**-distribution with ***N*-1** degree of freedom.

*

$X^{(N)}$

$x_1 \cdots x_N$

$E[T]=0$

$var[T] \approx 1$

$$T = \frac{\overbrace{mean(\{x\})}^{X^{(N)}} - \overbrace{popmean(\{X\})}^{E[X^{(N)}]}}{\underbrace{stderr(\{x\})}_{\approx std(X^{(N)})}}$$

N is the sample size

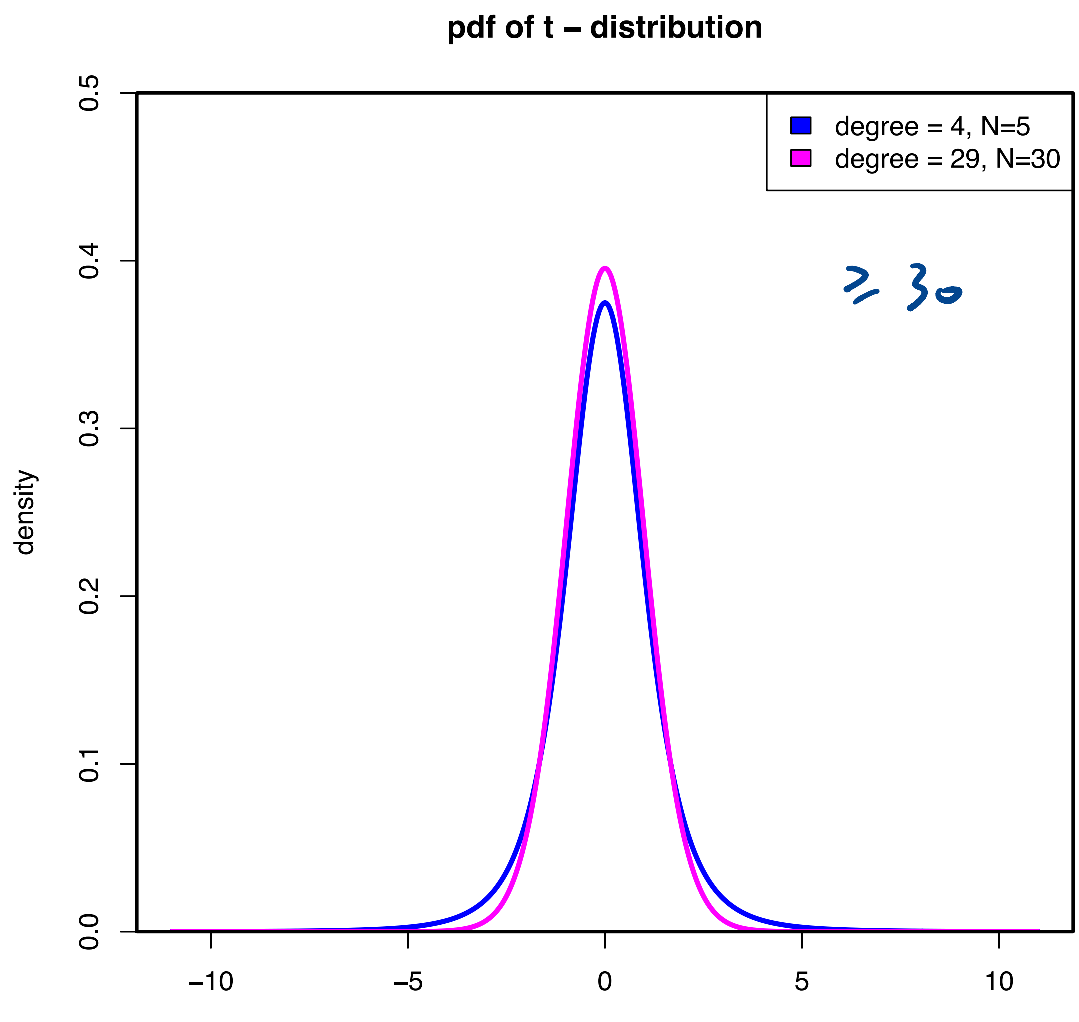
Degree of freedom is ***N***-1 due to this constraint: $\sum_i (x_i - mean(\{x\})) = 0$

# t-distribution is a family of distri. with different degrees of freedom

t-distribution with N=5 and N=30

Credit : wikipedia

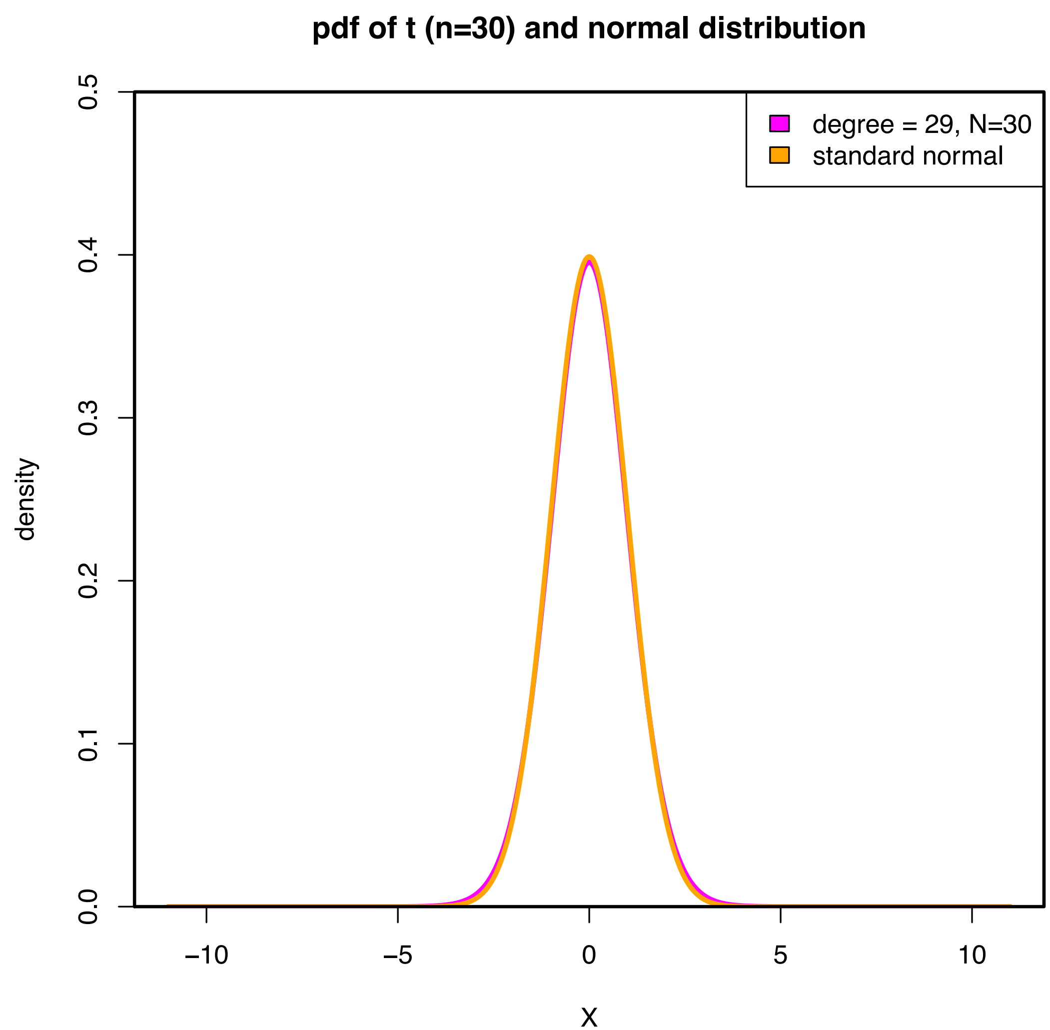
William Sealy Gosset 1876-1937



# When N=30, t-distribution is almost Normal

t-distribution looks very similar to normal when N=30.

So **N=30 is a rule of thumb to decide N is large or not**



# Assignments

- Read Chapter 7 of the textbook
- Next time: Bootstrap, Hypothesis tests
- Prepare for Midterm1

# Additional References


- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

# See you next time

*See you!*