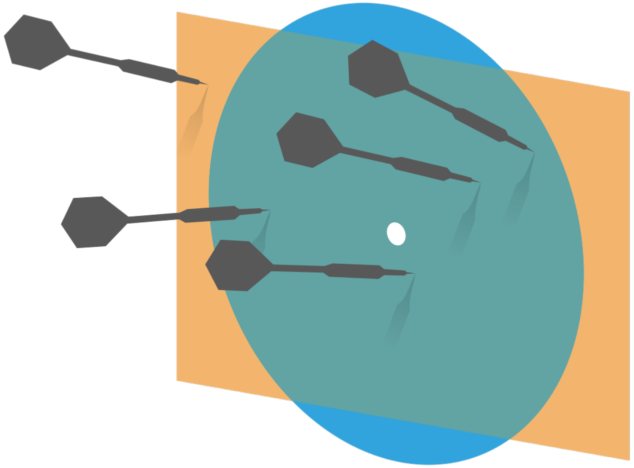
Probability and Statistics for Computer Science



Credit: wikipedia

"Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write." H. G. Wells

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 3.23.2021

Last time

* Hypothesis test

* Maximum Likelihood Estimation (MLE)

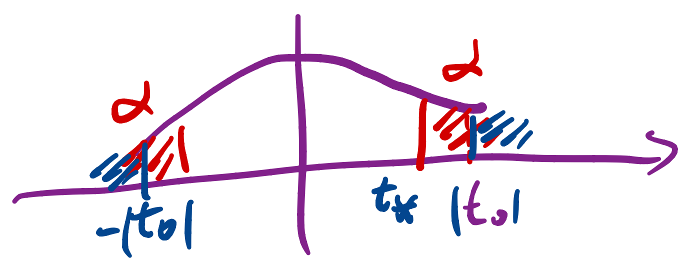
$2\alpha = 0.05$

$$T = \frac{X^{(N)} - \text{Assump}}{\text{stderr}} \qquad N \uparrow \infty$$

$$p = 1 - \int_{-|t_0|}^{|t_0|} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

if $p > 2\alpha$

if $p < 2\alpha$



Objectives

- More on Maximum likelihood Estimation (MLE) $P(D|\theta)$
- Bayesian Inference (MAP) $P(\theta|D)$

N

k

$p \rightarrow P(H)$

θ : is the p here

$\theta \rightarrow$ random varible

Maximum likelihood estimation (MLE)

- We write the probability of seeing the data D given parameter θ

$$L(\theta) = P(D|\theta)$$

- The **likelihood function** $L(\theta)$ is **not** a probability distribution

- The **maximum likelihood estimate (MLE)** of θ is

$$\hat{\theta} = arg\ \max_{\theta}\ L(\theta)$$

Likelihood function: binomial example

- Suppose we have a coin with unknown probability of θ coming up heads

- We toss it ***10*** times and observe ***7*** heads

D : N=10
K=7

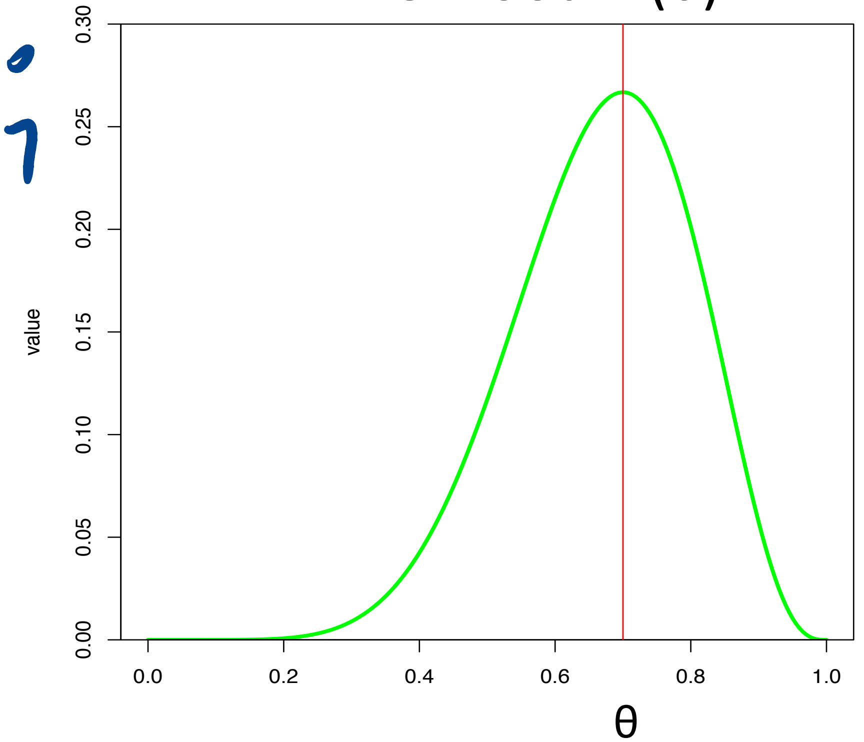
- The likelihood function is:

$$P(D|\theta) = \binom{10}{7}\theta^7(1-\theta)^3$$

- The MLE is

$$\hat{\theta} = 0.7$$

Likelihood ***L***(θ)



Q. What is the MLE of binomial N=12, k=7

A. 12!/7!/5!

B. 7/12

C. 5/12

D. 12/7

Q. What is the MLE of geometric k=7

A. 7

B. 1/7

C. other

Q. What is the MLE of Poisson $k_1=5$, $k_2=7$, $n=2$

A. 6

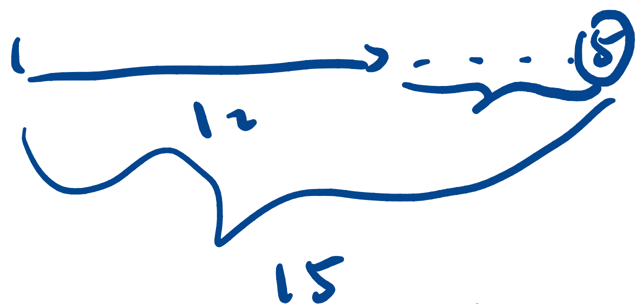
B. 35/2

C. 12

D. other

MLE Example

You find a 5-sided die and want to estimate its probability θ of coming up 5, you decided to roll it 12 times and then roll it until it comes up 5. You rolled 15 times altogether and found there were 3 times when the die came up 5. All rolls are independent. Write down the likelihood function L(θ).



$P(D \mid \theta)$

$D: D_1 \to N, k_1$

$D_2 \to k_2$

$k_1 = 3 - 1 = 2$

$N = 12$

$k_2 = 15 - 12 = 3$

MLE Example

$$L(\theta) = P(D|\theta) = P(D_1|\theta)\, P(D_2|\theta)$$

$$= \binom{N}{K_1} \theta^{K_1} (1-\theta)^{N-K_1} (1-\theta)^{K_2-1} \theta$$

$$N = 12 \quad K_1 = 2 \quad K_2 = 3$$

$$L(\theta) = \binom{12}{2} \theta^{3} (1-\theta)^{12}$$

$$\log L(\theta) = \log C + 3 \log \theta + 12 \log(1-\theta)$$

$$\frac{d \log L}{d\theta} = 0 + \frac{3}{\theta} - \frac{12}{1-\theta} = 0$$

$$\hat{\theta} = \frac{3}{15} = \frac{1}{5}$$

Drawbacks of MLE

- Maximizing some likelihood or log-likelihood function is mathematically hard
- If there are few data items, the MLE estimate maybe very unreliable
  - If we observe 3 heads in 10 coin tosses, should we accept that p(heads)= 0.3 ?
  - If we observe 0 heads in 2 coin tosses, should we accept that p(heads)= 0 ?

Bayesian inference

- In MLE, we maximized the likelihood function

$$L(\theta) = P(D|\theta)$$

- In Bayesian inference, we will maximize the **posterior**, which is the probability of the parameters **θ** given the observed data D.

$$P(\theta|D)$$

- Unlike $L(\theta)$, the posterior is a probability distribution

- The value of **θ** that maximizes $P(\theta|D)$ is called the **maximum a posterior (MAP)** estimate $\hat{\theta}$

The components of Bayesian Inference

- From Bayes rule

$$P(\theta \mid D) = \frac{\overset{L(\theta)}{P(D \mid \theta)}\, P(\theta)}{P(D)}$$

The components of Bayesian Inference

- From Bayes rule

$$P(\theta \mid D) = \frac{P(D \mid \theta)\, P(\theta)}{P(D)}$$

  - Prior, assumed distribution of **θ** before seeing data **D**
  - Likelihood function of **θ** seeing **D**
  - Total Probability seeing **D** --- P(**D**)
  - **Posterior**, distribution of **θ** given **D**

The usefulness of Bayesian inference

- From Bayes rule

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

- Bayesian inference allows us to include prior beliefs about θ in the prior $P(\theta)$, which is useful
  - When we have reasonable beliefs, such as a coin can not have P(heads) = 0
  - When there isn't much data
  - We get a distribution of the posterior, not just one maxima

Bayesian Inference: a discrete prior

- Suppose we have a coin of unknown probability θ of heads
  - We see 7 heads in 10 tosses (**D**)
  - We assume the prior about θ.

$$P(\theta) = \begin{cases} \frac{2}{3} & if\ \theta = 0.5 \\ \frac{1}{3} & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

  - We have this likelihood:

$$P(D|\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

  - What is the posterior $P(\theta|D)$?

Bayesian Inference: a discrete prior

- We see 7 heads in 10 tosses (**D**)
- We assume the prior about θ.

$$P(\theta) = \begin{cases} \frac{2}{3} & if\ \theta = 0.5 \\ \frac{1}{3} & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

- We have this likelihood:

$$P(D|\theta) = \binom{10}{7}\theta^7(1-\theta)^3$$

- What is the posterior $P(\theta|D)$?

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

Bayesian Inference: a discrete prior

- We see 7 heads in 10 tosses (**D**)
- We assume the prior about θ.

$$P(\theta) = \begin{cases} \frac{2}{3} & if\ \theta = 0.5 \\ \frac{1}{3} & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

- We have this likelihood:

$$P(D|\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

- What is the posterior $P(\theta|D)$?

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

$$P(D) = \sum_{\theta_i \in \theta} P(D|\theta_i)P(\theta_i)$$

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

$$P(\theta) = \begin{cases} \frac{2}{3} & \theta = 0.5 \\ \frac{1}{3} & \theta = 0.6 \\ 0 & \text{other} \end{cases}$$

$$P(D|\theta) = \binom{10}{7}\theta^7(1-\theta)^3$$

if $\theta = 0.6$

$$P(D) = \sum P(D|\theta_i)\cdot P(\theta_i)$$

$$= \binom{10}{7}0.5^7 0.5^3 \cdot \frac{2}{3} + \binom{10}{7}0.6^7 0.4^3 \cdot \frac{1}{3}$$

MLE $\hat{\theta} = 0.7$

$$P(\theta|D) = \begin{cases} 0.52 & \theta = 0.5 \\ 0.48 & \theta = 0.6 \\ 0 & \text{other} \end{cases}$$

which θ maximize $P(\theta|D)$: $\hat{\theta} = 0.5$

Bayesian Inference: a discrete prior

- We see 7 heads in 10 tosses (**D**)
- We assume the prior about θ.

$$P(\theta) = \begin{cases} \frac{2}{3} & if\ \theta = 0.5 \\ \frac{1}{3} & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

- We have this likelihood:

$$P(D|\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

- What is the posterior $P(\theta|D)$?

$$P(\theta|D) = \begin{cases} 0.52 & if\ \theta = 0.5 \\ 0.48 & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

MAP estimate=?

Bayesian Inference: a discrete prior

- We see 7 heads in 10 tosses (**D**)
- We assume the prior about θ.

$$P(\theta) = \begin{cases} \frac{2}{3} & if\ \theta = 0.5 \\ \frac{1}{3} & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

- We have this likelihood:

$$P(D|\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

- What is the posterior $P(\theta|D)$?

$$P(\theta|D) = \begin{cases} 0.52 & if\ \theta = 0.5 \\ 0.48 & if\ \theta = 0.6 \\ 0 & otherwise \end{cases}$$

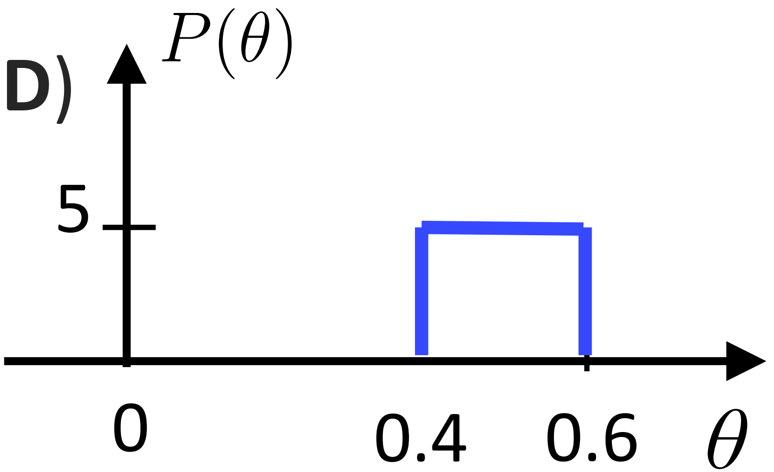
MAP $\hat{\theta}$ **=0.5**

Biased by the prior

Bayesian Inference: a continuous prior

- Suppose we have a coin of unknown probability θ of heads
- We see 7 heads in 10 tosses (**D**)
- We assume

$$P(\theta) = \begin{cases} 5 & if\ \theta \in [0.4, 0.6] \\ 0 & if\ \theta \notin [0.4, 0.6] \end{cases}$$

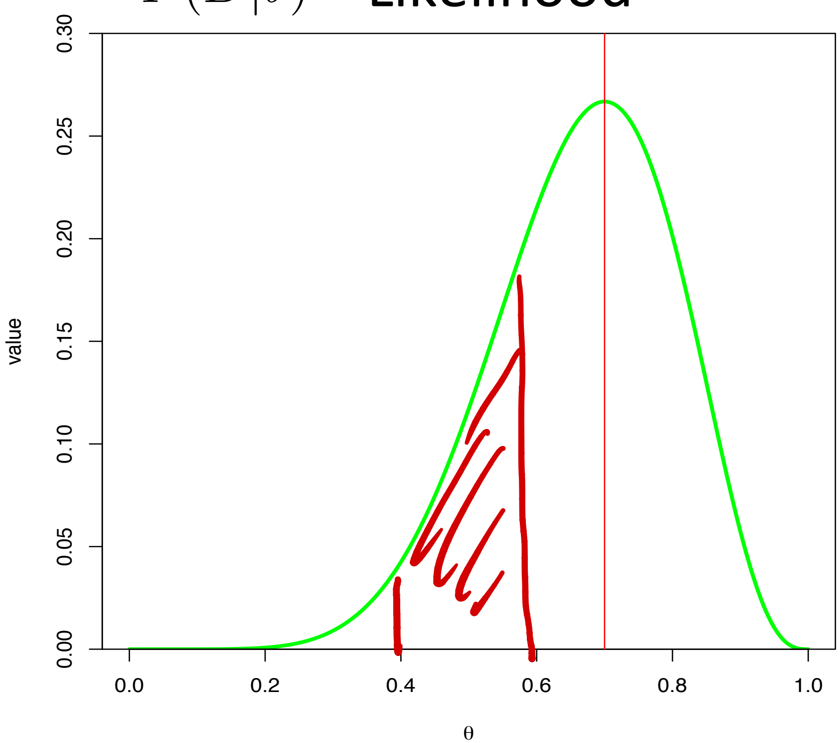


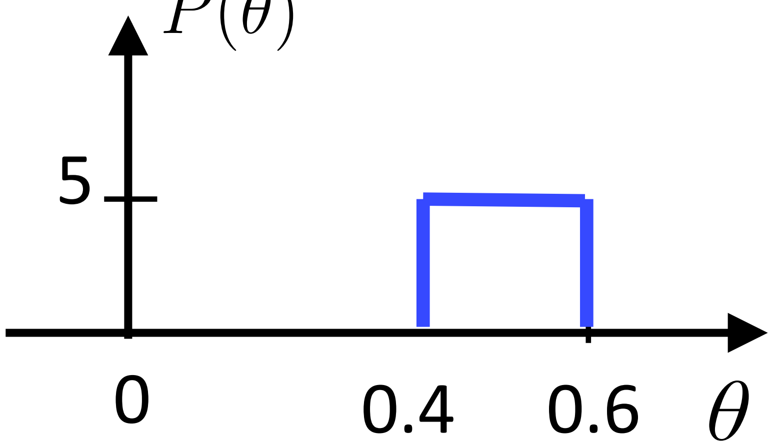
- What is the posterior $P(\theta|D)$?

Bayesian Inference: a continuous prior

- What is the posterior $P(\theta|D)$?

$P(D|\theta)$ = Likelihood





$$P(\theta) = \begin{cases} 5 & if \ \theta \in [0.4, 0.6] \\ 0 & if \ \theta \notin [0.4, 0.6] \end{cases}$$

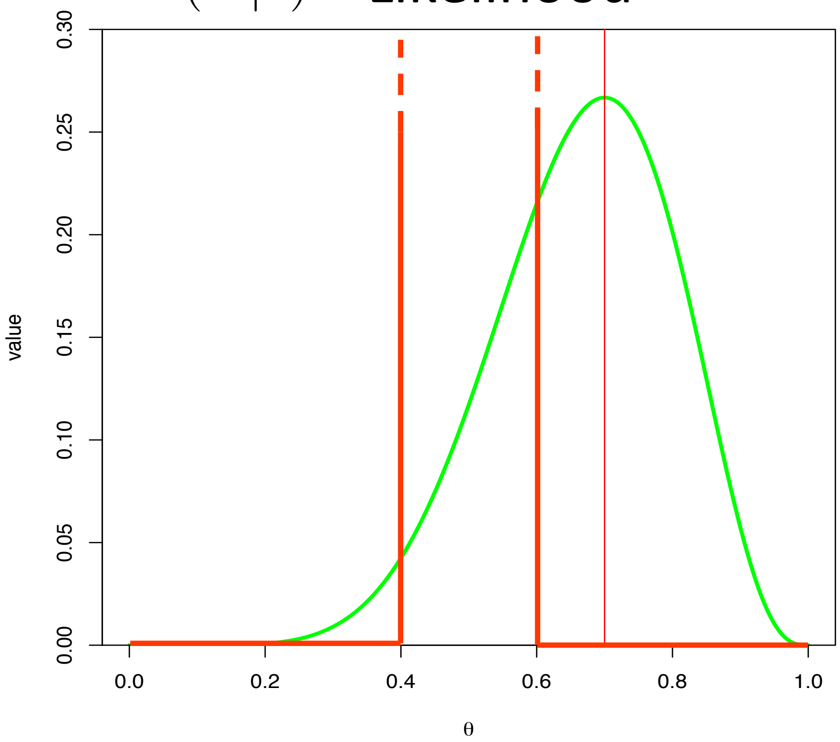
$$P(\theta|D) \propto P(D|\theta)P(\theta)$$

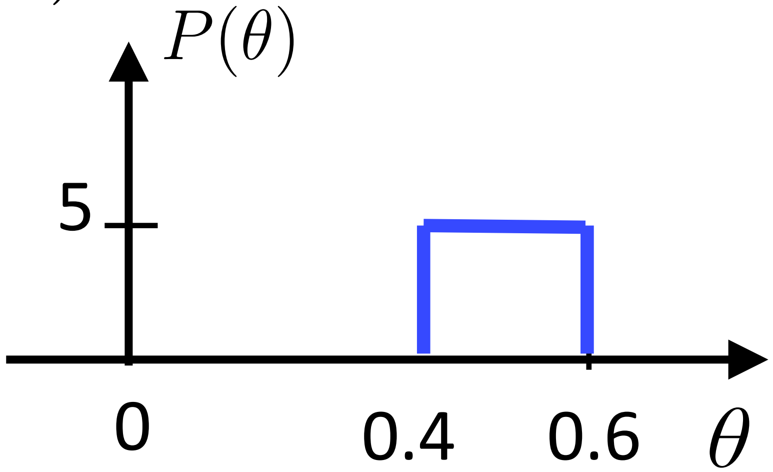
$\hat{\theta} = 0.6$

Bayesian Inference: a continuous prior

- What is the posterior $P(\theta|D)$?

$P(D|\theta)$ = Likelihood





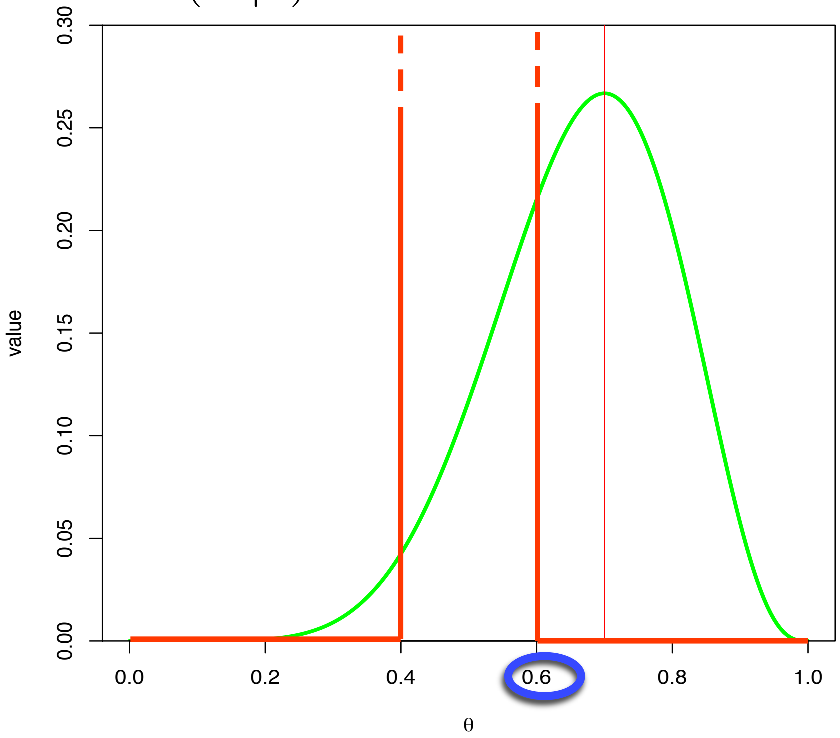
$$P(\theta) = \begin{cases} 5 & if\ \theta \in [0.4, 0.6] \\ 0 & if\ \theta \notin [0.4, 0.6] \end{cases}$$

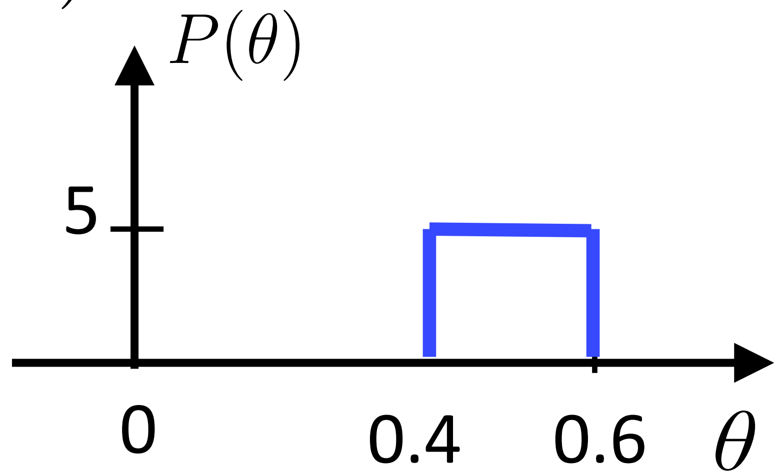
$$P(\theta|D) \propto P(D|\theta)P(\theta)$$

Bayesian Inference: a continuous prior

- What is the posterior $P(\theta|D)$?

$P(D|\theta)$ = Likelihood





$$P(\theta) = \begin{cases} 5 & if\ \theta \in [0.4, 0.6] \\ 0 & if\ \theta \notin [0.4, 0.6] \end{cases}$$

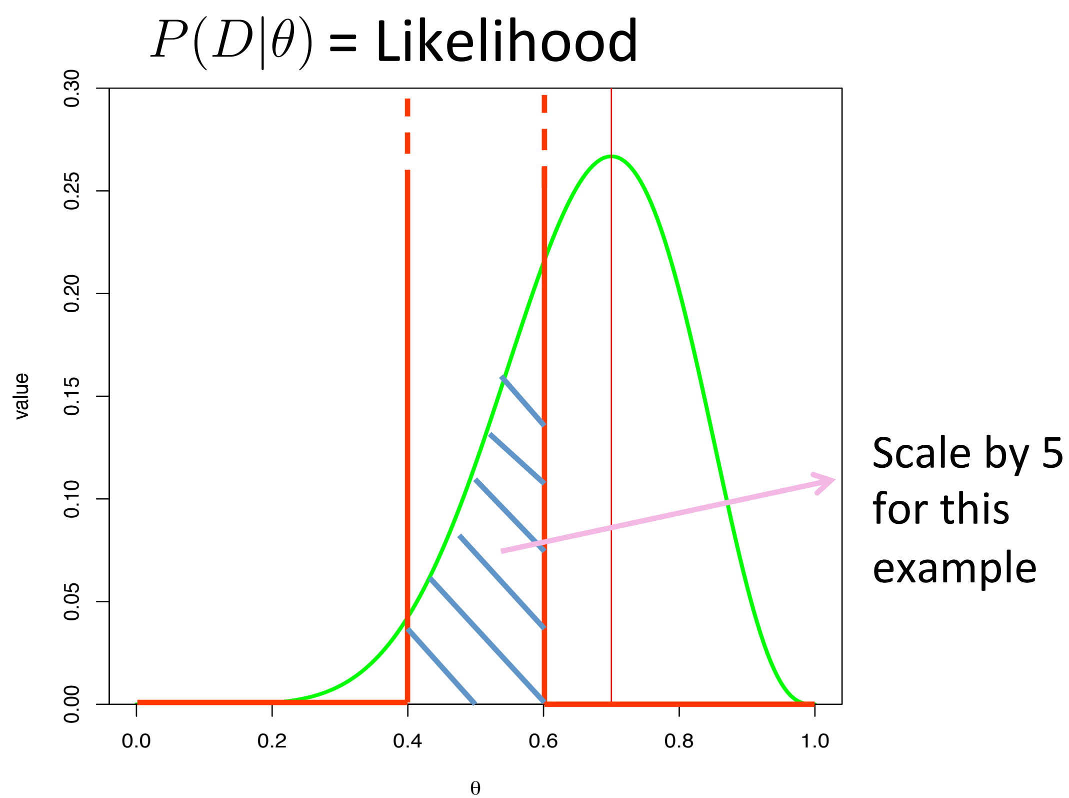
$$P(\theta|D) \propto P(D|\theta)P(\theta)$$

MAP $\hat{\theta}$ **=0.6**

The constant in the Bayesian inference

$$P(D) = \int_\theta P(D|\theta)P(\theta)d\theta$$

- It's not always possible to calculating P(D) in closed form.
- There are a lot of approximation methods.



Drawbacks of Bayesian inference

- Maximizing some posteriors $P(\theta|D)$ is difficult
- Some choices of prior $P(\theta)$ can overwhelm any data observed.
- It's hard to justify a choice of prior

The concept of conjugacy

- For a given likelihood function $P(D|\theta)$, a prior $P(\theta)$ is its conjugate prior if it has the following properties:
  - $P(\theta)$ belongs to a family of distributions that are expressive
  - The posterior $P(\theta|D) \propto P(D|\theta)P(\theta)$ belongs to the same family of distribution as the prior $P(\theta)$
  - The posterior $P(\theta|D)$ is easy to maximize
- For example, a conjugate prior for binomial likelihood function is Beta distribution

Beta distribution

- A distribution is Beta distribution if it has the following pdf: $$P(\theta) = K(\alpha, \beta)\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

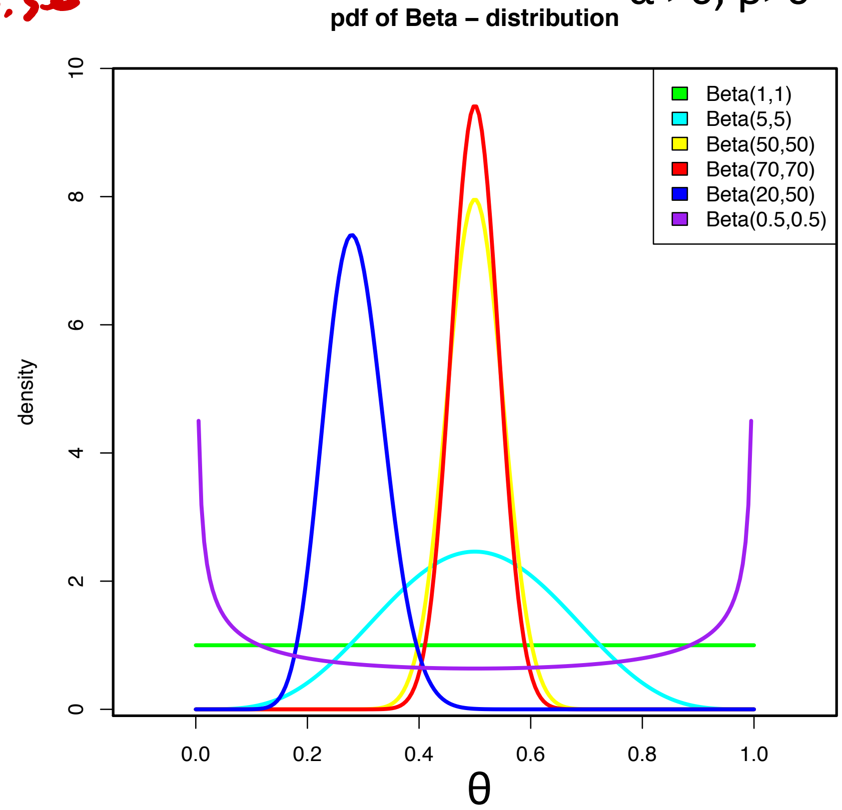
  = 0 O.W. → other wise

  $0 \leq \Theta \leq 1$

  $\alpha > 0, \beta > 0$

$$K(\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

- Is an expressive family of distributions
- $Beta(\alpha = 1, \beta = 1)$ is uniform



Q. Beta distribution is a continuous probability distribution

A. TRUE

B. FALSE

$$p(\theta|D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

Beta distribution as the conjugate prior for Binomial likelihood

$p(D|\theta)\,p(\theta)$

- The likelihood is Binomial (*N*, *k*)

$$P(D|\theta) = \binom{N}{k} \theta^k (1-\theta)^{N-k}$$

- The Beta distribution is used as the prior

$$P(\theta) = K(\alpha,\beta)\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$\theta \in [0,1]$

$\hat{\beta} = \beta + N - k$

$\hat{\alpha}, \hat{\beta}$?

- So $P(\theta|D) \propto \theta^{\alpha+k-1}(1-\theta)^{\beta+N-k-1}$

$\hat{\alpha} = \alpha + k$

if $P(\theta|D) \sim Beta(\hat{\alpha}, \hat{\beta})$

- Then the posterior is $Beta(\alpha+k, \beta+N-k)$

$$P(\theta|D) = K(\alpha+k,\beta+N-k)\theta^{\alpha+k-1}(1-\theta)^{\beta+N-k-1}$$

The update of Bayesian posterior

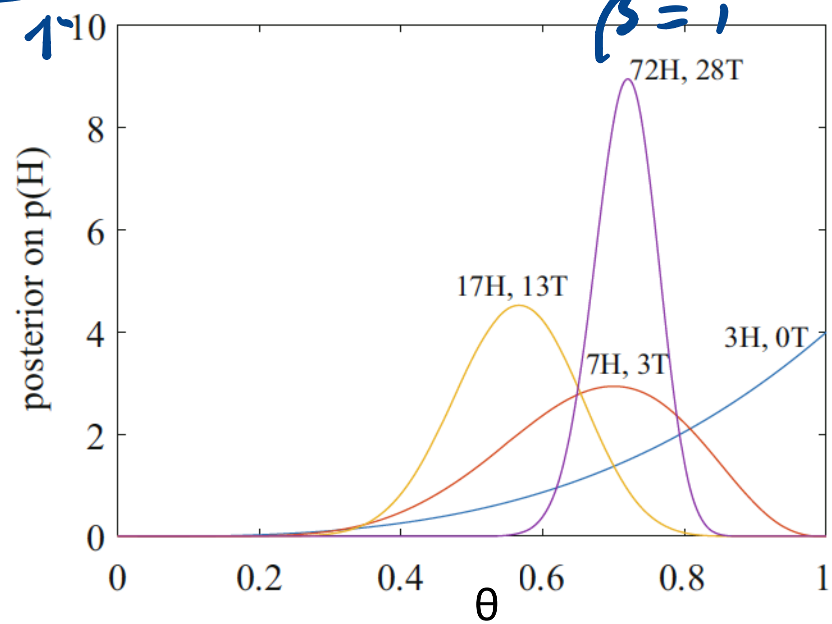
- Since the posterior is in the same family as the conjugate prior, the posterior can be used as a new prior if more data is observed.

$p(\theta) = \begin{cases} 1 & \text{if } \theta \in [0,1] \\ 0 & \text{ow} \end{cases}$

- Suppose we start with a uniform prior on the probability θ of heads
  - Then we see 3H 0T
  - Then we see 4H 3T for 7H 3T in total
  - Then we see 10H 10T for 17H 13T in total
  - Then we see 55H 15T for 72H 28T in total

$\alpha = 1$

$\beta = 1$

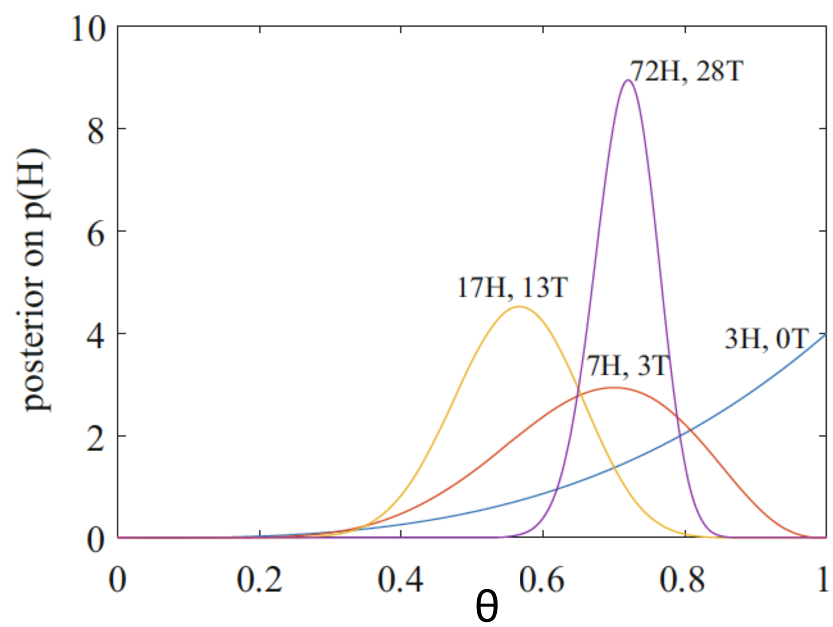


The update of Bayesian posterior

- Since the posterior is in the same family as the conjugate prior, the posterior can be used as a new prior if more data is observed.

- Suppose we start with a uniform prior on the probability θ of heads

N	k	α	β
		1	1
3	0	1	4
10	7	8	7
30	17	25	20
100	72	97	48



Simulation of the update of Bayesian posterior

https://seeing-theory.brown.edu/bayesian-inference/index.html

Maximize the Bayesian posterior (MAP)

- The posterior of the previous example is

α β are prior para.

$$P(\theta|D) = K(\alpha + k, \beta + N - k)\theta^{\alpha+k-1}(1-\theta)^{\beta+N-k-1}$$

$P(\theta) \sim Beta(\alpha, \beta)$

- Differentiating and setting to 0 gives the MAP estimate

$$\hat{\theta} = \frac{\alpha - 1 + k}{\alpha + \beta - 2 + N}$$

Conjugate prior for other likelihood functions

- If the likelihood is Bernoulli or geometric, the conjugate prior is Beta
- If the likelihood is Poisson or Exponential, the conjugate prior is Gamma
- If the likelihood is normal with known variance, the conjugate prior is normal

Assignments

- Finish Chapter 9 of the textbook
- Next time: Covariance matrix, PCA

Additional References

- Robert V. Hogg, Elliot A. Tanis and Dale L. Zimmerman. “Probability and Statistical Inference”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!