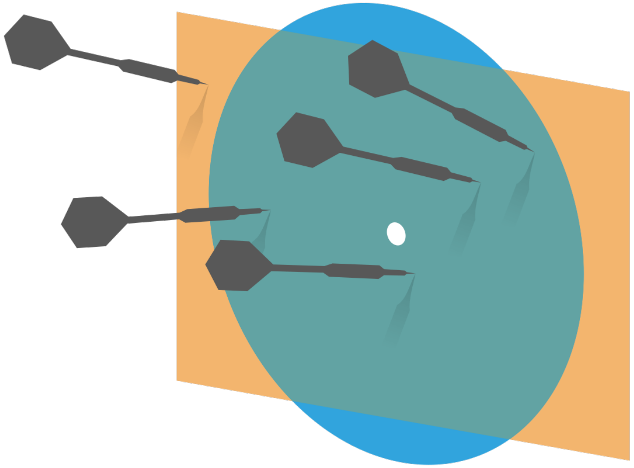
Probability and Statistics for Computer Science



Credit: wikipedia

"Probabilistic analysis is mathematical, but intuition dominates and guides the math" – Prof. Dimitri Bertsekas

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 2.4.2021

Homework (I)

- Due 2/4 today at 11:59pm
- There is one optional problem with extra 5 points. (Won't be in exams)

Last time

Correlation Coeffient (corr).

Prediction using corr.

Warm up

A game of chance.

Warm up (II)

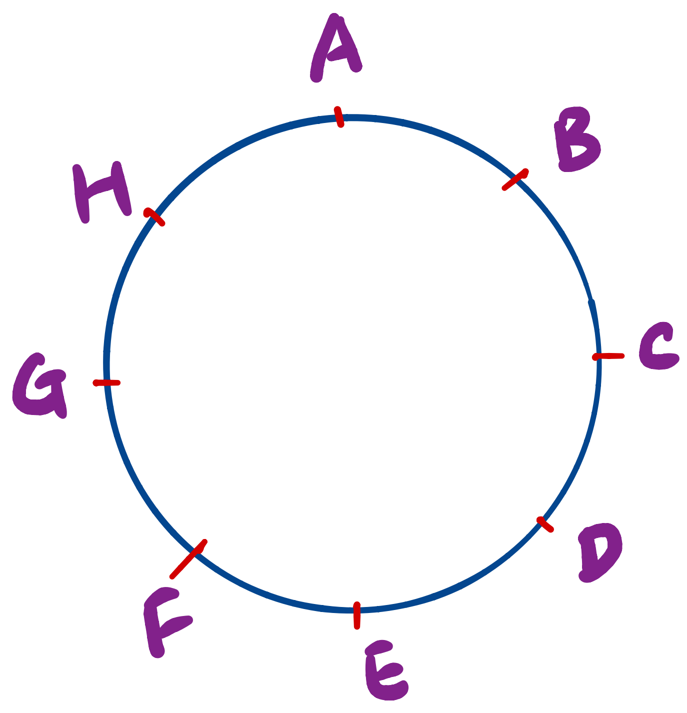
- Fill the blanks:

"I am an avid vegetarian and I enjoy eating all day long, people admire my appetite and like to watch me eat." I am a PANDA_ _ _ _ _. How many ways are there to rearrange these 5 letters? $\frac{5!}{2!}$_______. If you draw 2 letters from them, how many outcomes (order matters) are there that are without "a"?

$\underline{3 \times 2}$

How many arrangements?

8 people to sit by a round table.



7!

A B . . . (line up)

8!

A B C D E F G H

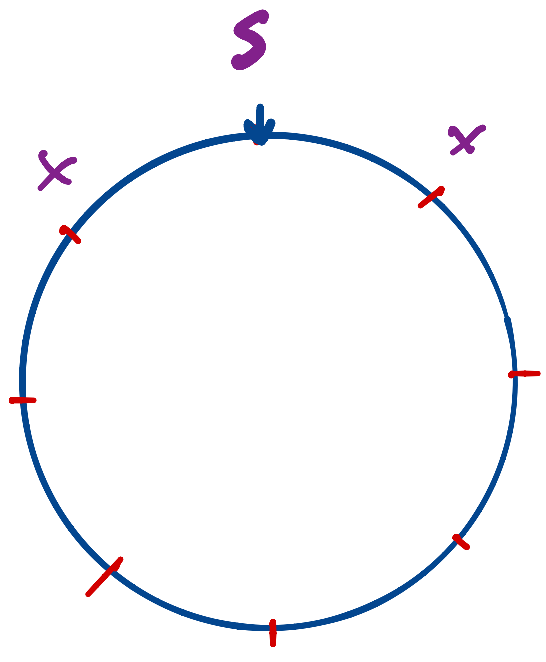
→ B C D E . . . H A

. . .

$$\frac{8!}{8} = 7!$$

How much chance ?

One student (S) and the best friend (B) get to sit together.



$$\frac{2 \times 6!}{7!} = \frac{2}{7}$$

Objectives

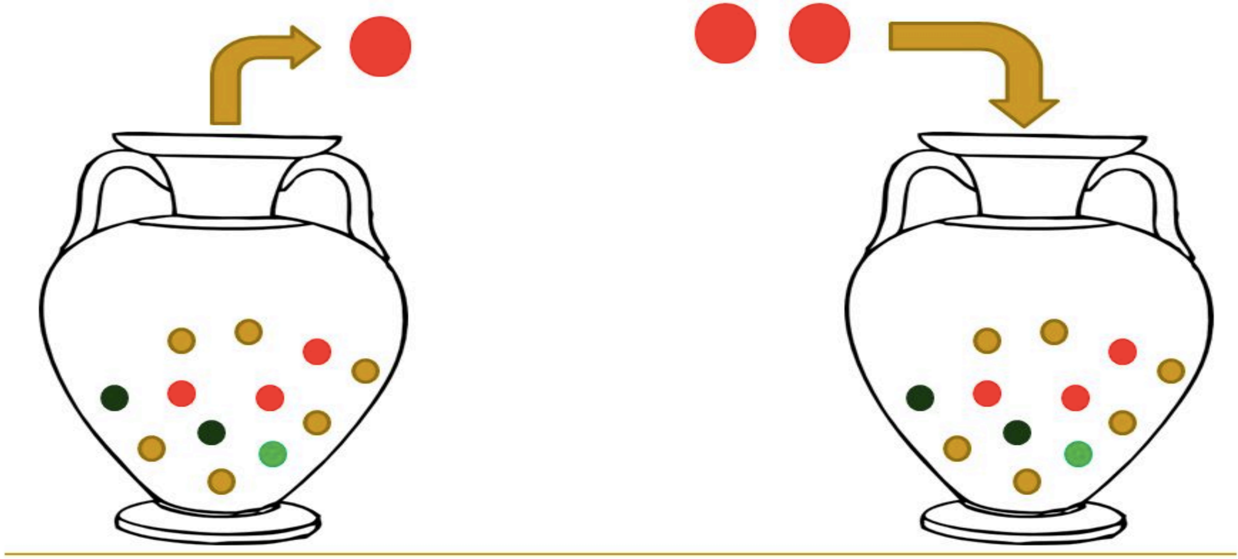
- Probability a first look
  - Outcome and Sample Space
  - Event
  - Probability

    Probability axioms & Properties
  - Calculating probability

Outcome

- An outcome **A** is a possible result of a random repeatable experiment

Random: uncertain, Nondeter-ministic, …



Sample space

- The Sample Space, **Ω,** is the set of all possible outcomes associated with the experiment
- Discrete or Continuous

Sample Space example (1)

- Experiment: we roll a 4sided-die twice
- **Discrete** Sample space:

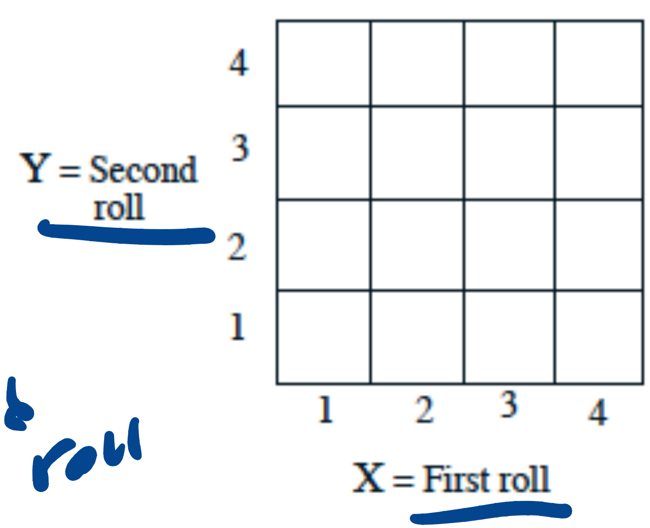
List

{(1,1), (1,2)....}

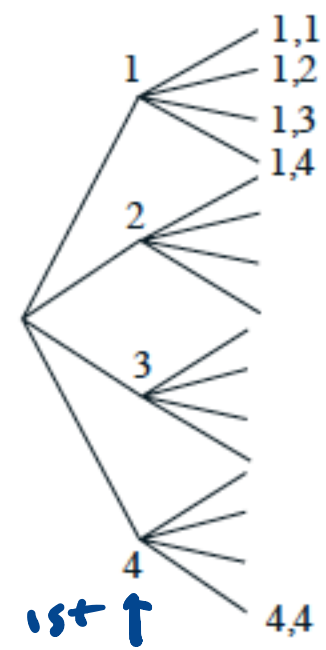
1st roll

2nd roll

Table



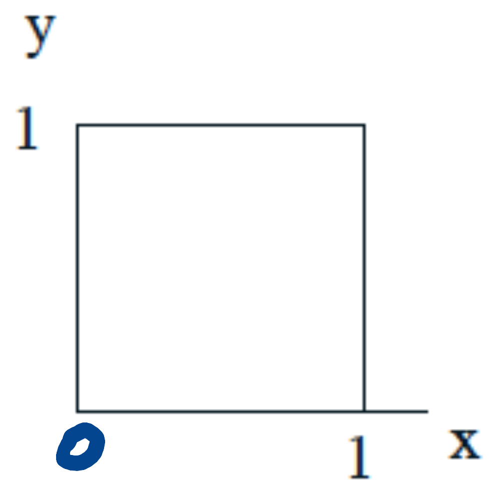
Tree



Sample Space example (2)

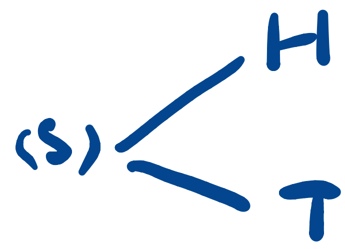
- Experiment: Romeo and Juliet's date
- **Continuous** Sample space:

$$\Omega = \{(x, y) \mid \ 0 \leq x, y \leq 1\}$$

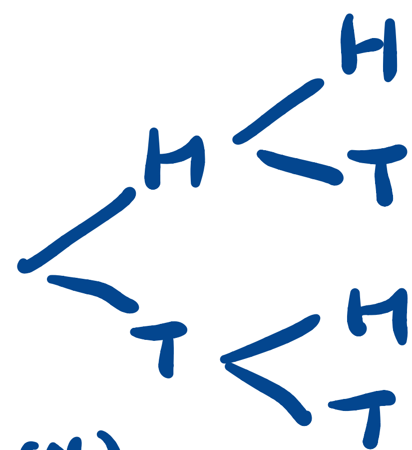


Sample Space depends on experiment (3)

- Different coin tosses
  - Toss a fair coin {H, T}



  - Toss a fair coin twice



{HH, HT, TH, TT}

  - Toss until a head appears

H, TH, TTH . . . . TT··TH

(N)

Sample Space depends on experiment (4)

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**?

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**?

Q.

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **with replacement**? What is the number of unique outcomes in the sample space?

A. 5 B. 7 C. 9

$3 \times 3 = 9$

Q.

- Drawing 2 socks one at a time from a bag containing 1 blue sock, 1 orange sock and 1 white sock **without replacement**? What is the number of unique outcomes in the sample space?

A. 5 B. 6 C. 9

$3 \times 2 = 6$

Sample Space in real life

- Possible outrages of a power network
- Possible mutations in a gene
- A bus' arriving time

Event

- An event **E** is a subset of the sample space Ω
- So an event is a set of outcomes that is a subset of Ω, ie.
  - Zero outcome ϕ
  - One outcome $\{A\}$
  - Several outcomes $\{A, B, C\}$
  - All outcomes $E = \Omega$

The same experiment may have different events

- When two coins are tossed
  - Both coins come up the same? $E_1 = \{HH, TT\}$
  - At least one head comes up?

$E_2 : \{HH, HT, TH\}$

Some experiment may never end

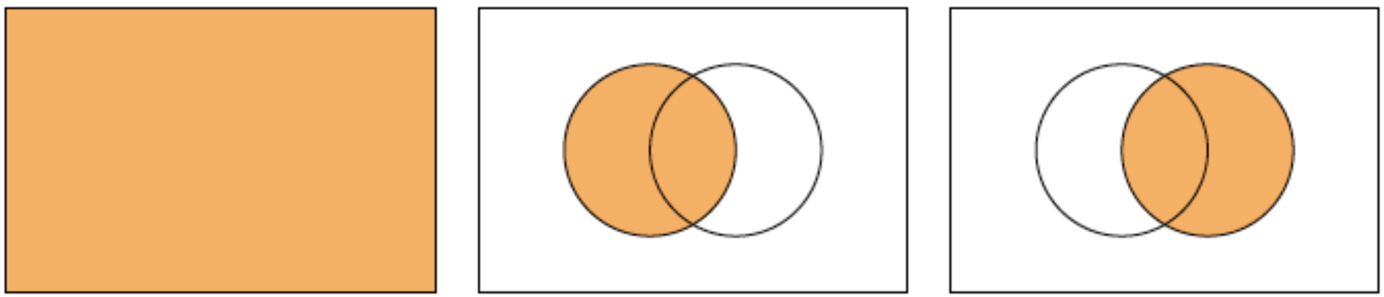
- Experiment: Tossing a coin until a head appears

  - **E**: Coin is tossed at least 3 times

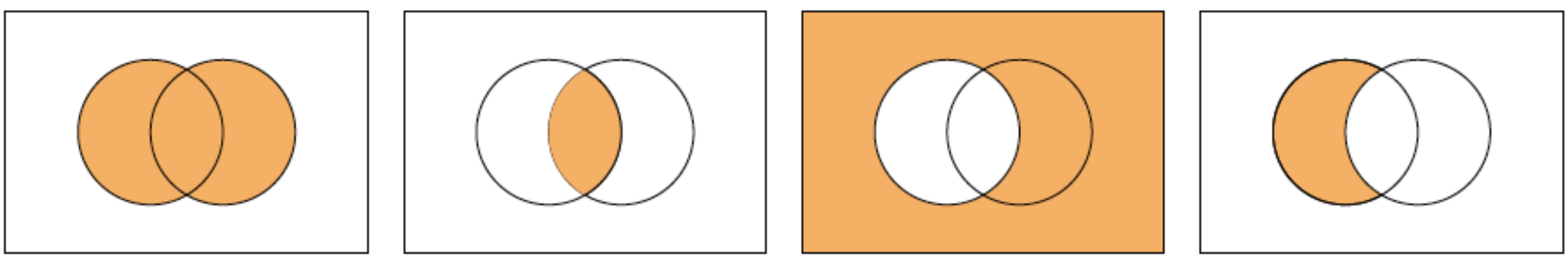
    This event includes infinite # of outcomes

E: {TTH, TTTH, . . . }

Venn Diagrams of events as sets



Ω E_1 E_2



$E_1 \cup E_2$ $E_1 \cap E_2$ ${E_1}^c$ $E_1 - E_2$

Combining events

- Say we roll a six-sided die. Let Ω ?

$$E_1 = \{1, 2, 5\} \; and \; E_2 = \{2, 4, 6\}$$

$\Omega = \{1, 2, 3, 4, 5, 6\}$

- What is $E_1 \cup E_2 = \{1, 2, 5, 4, 6\}$
- What is $E_1 \cap E_2 = \{2\}$
- What is $E_1 - E_2 = \{1, 5\}$
- What is $E_1^c = \Omega - E_1 = \{3, 4, 6\}$

Frequency Interpretation of Probability

- Given an experiment with an outcome **A**, we can calculate the probability of **A** by repeating the experiment over and over

$$P(A) = \lim_{N->\infty} \frac{number\ of\ time\ A\ occurs}{N}$$

- So,

$$0 \leq P(A) \leq 1$$

$$\sum_{A_i \in \Omega} P(A_i) = 1$$

Axiomatic Definition of Probability

- A probability function is any function P that maps sets to real number and satisfies the following **three** axioms:

  1) Probability of any event E is non-negative

$$P(E) \geq 0$$

  2) Every experiment has an outcome

$$P(\Omega) = 1$$

Axiomatic Definition of Probability

Mutually exclusive

3) The probability of disjoint events is additive

$$P(E_1 \cup E_2 \ \cup \ldots \cup \ E_N) = \sum_{i=1}^{N} P(E_i)$$

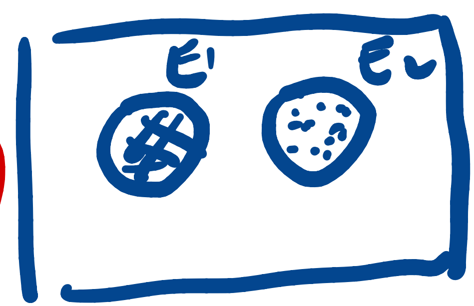
$$if \ E_i \cap E_j = \emptyset \ for \ all \ i \neq j$$

$E_1 \cap E_2 = \phi$

$\{1, 2\}$ $\{3, 4\}$

$E_3 = E_1 \cup E_2$

$P(E_3) = P(E_1) + P(E_2)$



Q.

- Toss a coin 3 times

The event "exactly 2 heads appears" and "exactly 2 tails appears" are disjoint.

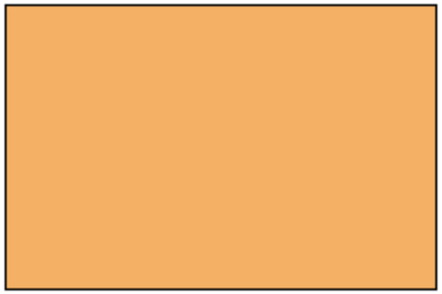
A. True

B. False

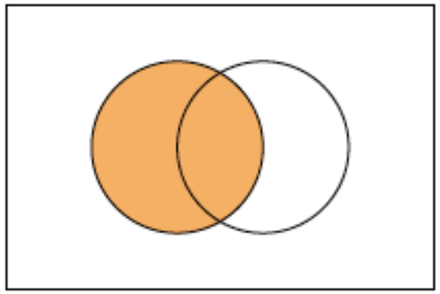
{HHT THH HTH}

{HTT THT TTH}

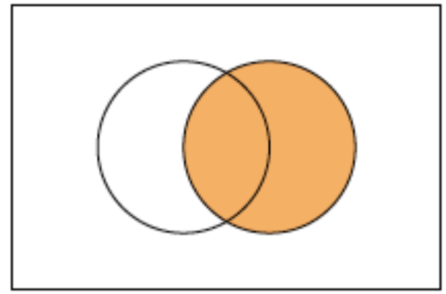
Venn Diagrams of events as sets



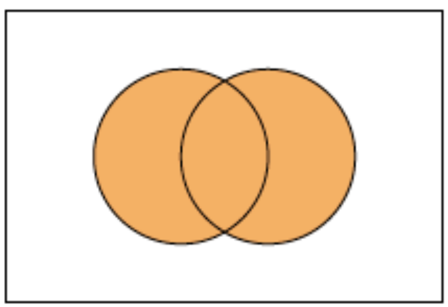
Ω



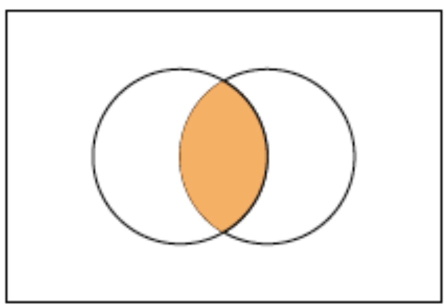
E_1



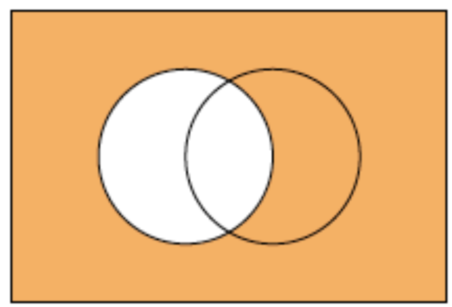
E_2



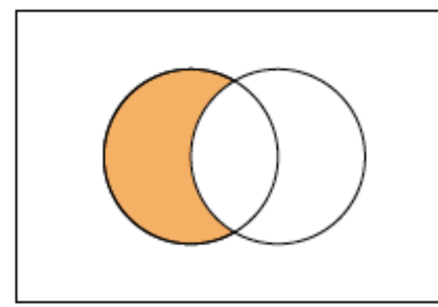
$E_1 \cup E_2$



$E_1 \cap E_2$



${E_1}^c$

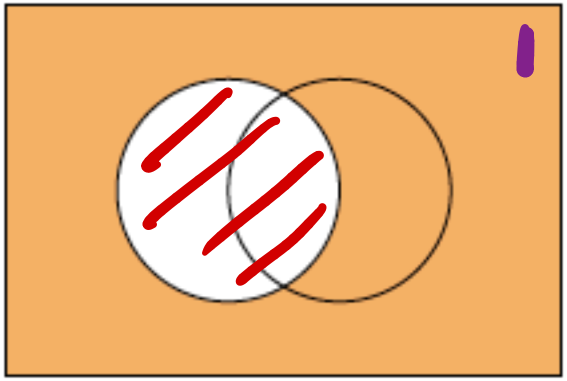


$E_1 - E_2$

Properties of probability

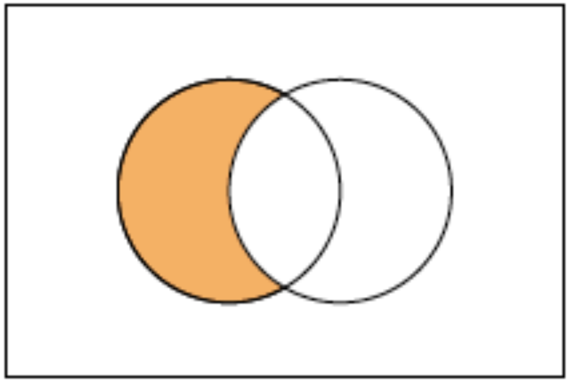
- The complement

$$P(E^c) = 1 - P(E)$$



- The difference

$$P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$$

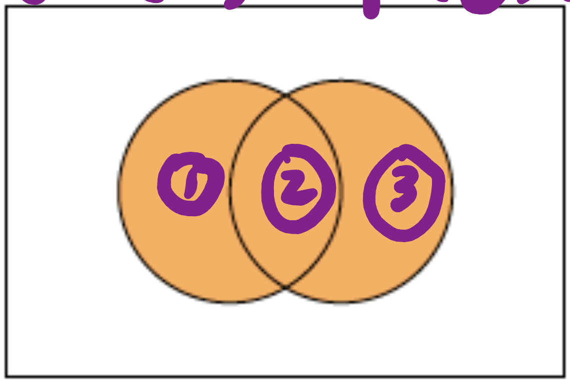


Properties of probability

- The union

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$P(E_1 \cup E_2) = P(1) + P(2) + P(3)$



axiom #3

$P(E_1) = P(1) + P(2)$

$P(E_2) = P(2) + P(3)$

- The union of multiple E

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) + P(E_1 \cap E_2 \cap E_3)$$

$(-1)^{k-1}$

The Calculation of Probability

- Discrete countable finite event
- Discrete countable infinite event
- Continuous event

Counting to determine probability of countable finite event

- From the last axiom, the probability of event **E** is the sum of probabilities of the disjoint outcomes

$$P(E) = \sum_{A_i \in E} P(A_i)$$

- If the outcomes are atomic and have equal probability,

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ \Omega}$$

Probability using counting: (1)

- Tossing a fair coin twice:
  - Prob. that it appears the same?
  - Prob. that at least one head appears?

Probability using counting: (2)

- 4 rolls of a 5-sided die:

  E: they all give different numbers

  - Number of outcomes that make the event happen: $5 \times 4 \times 3 \times 2$

  - Number of outcomes in the sample space

    5^4

  - Probability: $\frac{|E|}{|S|} = \frac{5 \times 4 \times 3 \times 2}{5^4}$

Probability using counting: (2)

- What about N-1 rolls of a N-sided die?

  E: they all give different numbers

  - Number of outcomes that make the event happen:

  - Number of outcomes in the sample space

  - Probability:

Probability by reasoning with the complement property

- If $P(E^c)$ is easier to calculate

$$P(E) = 1 - P(E^c)$$

Probability by reasoning with the complement property

- A person is taking a test with **N** true or false questions, and the chance he/she answers any question right is 50%, what's probability the person answers **at least** one question right?

E^c: None is correct

$|E^c| = ?\quad 1$

$|\Omega| = \quad 2^N$

$P(E^c) = \frac{1}{2^N}$

$P(E) = 1 - \frac{1}{2^N}$

Probability by reasoning with the union property

- If E is either E1 or E2

$$P(E) = P(E_1 \cup E_2) =$$

$$P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Probability by reasoning with the properties (2)

- A person may ride a bike on any day of the year equally. What's the probability that he/she rides on a Sunday or on 15th of a month? In 2021,

E: ride on Sun or 15th
E_1 or E_2

1 day both 15th & Sun.

$P(E) = ?$

52 Sundays

$= P(E_1 \cup E_2)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

12 months

$= \frac{52}{365} + \frac{12}{365} - \frac{1}{365}$

Counting may not work

- This is one important reason to use the method of reasoning with properties

What if the event has infinite outcomes

- Tossing a coin until head appears
  - Coin is tossed at least 3 times

    This event includes infinite # of outcomes.

    And the outcomes don't have equal probability.

TTH, TTTH, TTTTH….

Assignments

- Do Module Week2, Quiz2

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!