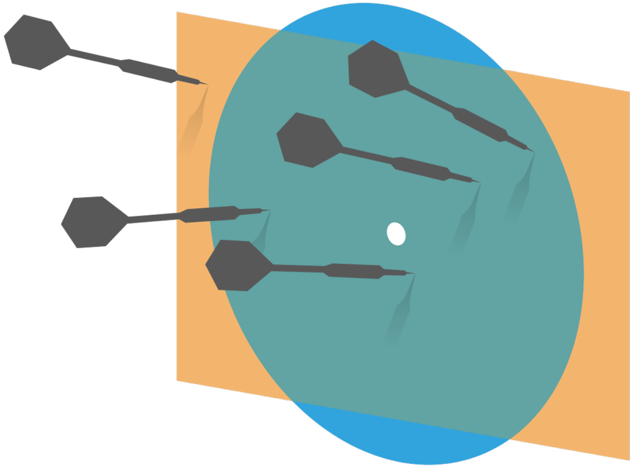
Probability and Statistics for Computer Science



Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 2.9.2021

Last time

- Probability a first look
  - Outcome and Sample Space
  - Event
  - Probability

    Probability axioms & Properties
  - Calculating probability

Objectives

- Probability
  - More probability calculation
  - Conditional Probability
  - Independence

Senate Committee problem

The United States Senate contains **two** senators from each of the **50** states. If a committee of eight senators is selected at random, what is the probability that it will contain at least one of the two senators from **IL**?

Probability: Birthday problem

- Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.

Conditional Probability

- Motivation of conditional probability

Conditional Probability

- Example:

An insurance company knows in a population of 100 thousands females, 89.835% expect to live to age 60, while 57.062% can expect to live to 80. Given a woman at the age of 60, what is the probability that she lives to 80?

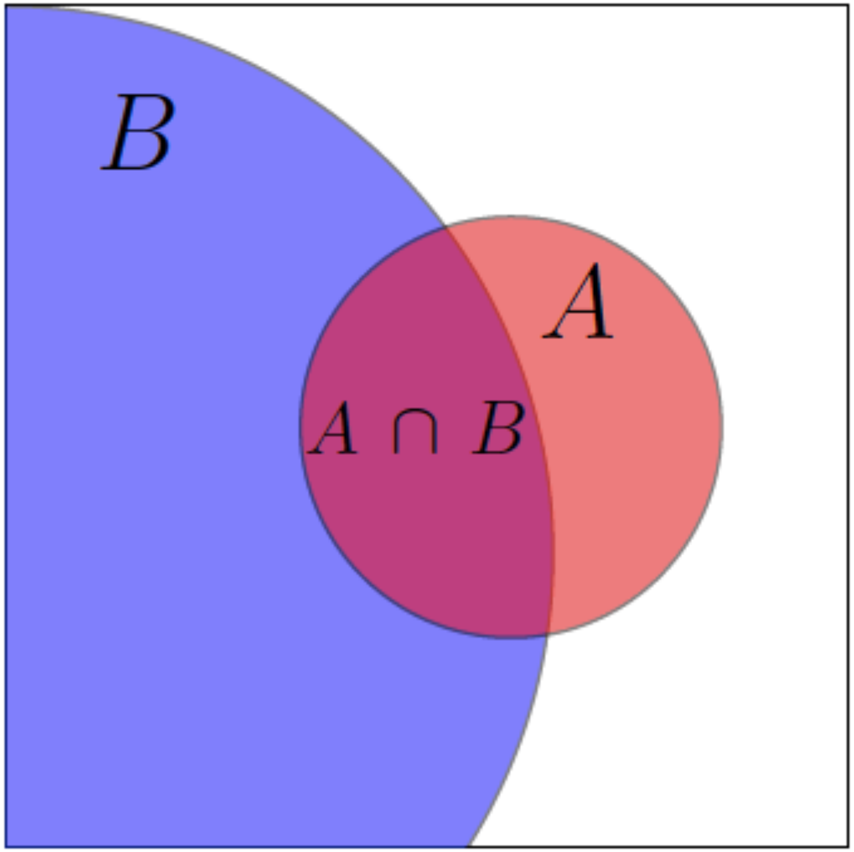
Conditional Probability

- Given the condition she is 60 already, the size of the sample space for the outcomes has changed to

  89,835 instead of 100,000

Conditional Probability

- The probability of ***A*** given ***B***



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The "Size" analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom

Conditional Probability

A : a woman lives to 80

$$P(A|B) = \frac{57,062}{89,835} = 0.6352$$

B : a woman is at 60 now

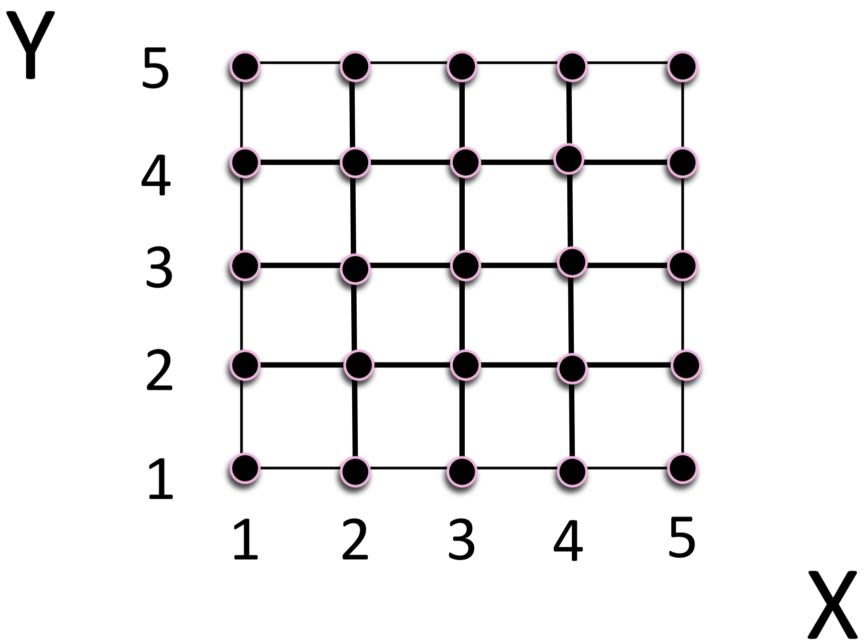
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

While $P(A) = \frac{57,062}{100,000} = 0.57062$

Conditional Probability: die example

Throw 5-sided fair die twice.

$$A : max(X, Y) = 4$$
$$B : min(X, Y) = 2$$



$$P(A|B) = ?$$

Conditional probability, that is?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

Multiplication rule using conditional probability

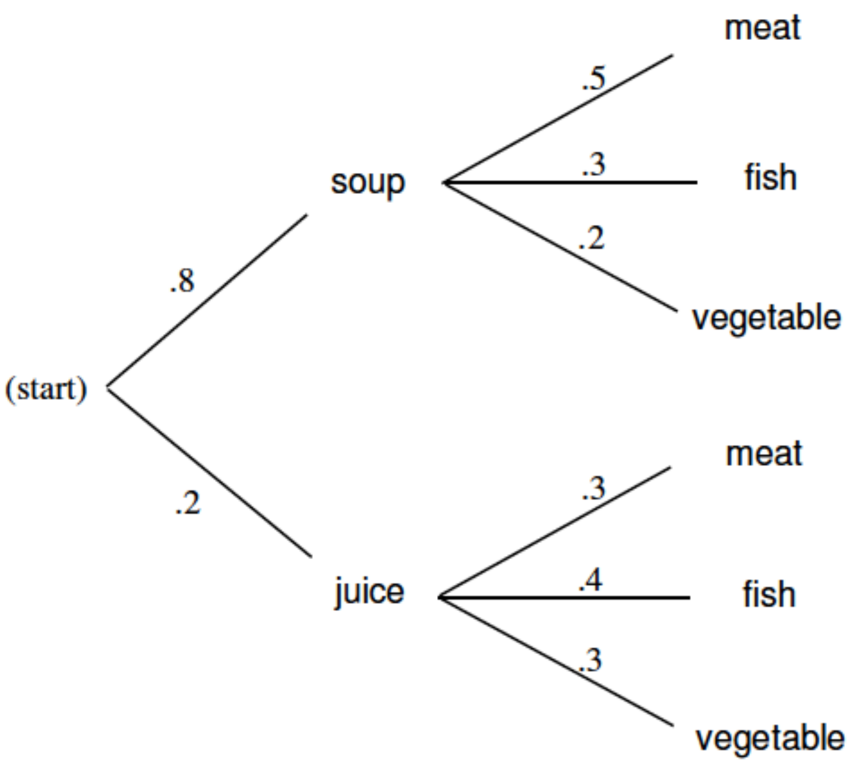
- Joint event

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow \boxed{P(A \cap B) = P(A|B)P(B)}$$

Multiplication using conditional probability

$$P(A \cap B) = P(A|B)P(B)$$



$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Symmetry of joint event in terms of conditional prob.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) \neq 0$$

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$\Rightarrow P(B \cap A) = P(B|A)P(A)$$

Symmetry of joint event in terms of conditional prob.

$$\because P(B \cap A) = P(A \cap B)$$

$$\Rightarrow$$

$$P(A|B)P(B) = P(B|A)P(A)$$

The famous **Bayes** rule

$$P(A|B)P(B) = P(B|A)P(A)$$

$$\Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Thomas Bayes (1701-1761)

Bayes rule: lemon cars

There are two car factories, **A** and **B**, that supply the same dealer. Factory **A** produced **1000** cars, of which **10** were lemons. Factory **B** produced **2** cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory **B**?

Bayes rule: lemon cars

There are two car factories, A and B, that supply the same dealer. Factory A produced 1000 cars, of which 10 were lemons. Factory B produced 2 cars and both were lemons. You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

$$P(B|L) = \frac{P(L|B)P(B)}{P(L)}$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

$$P(A|L) = ?$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

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$$P(A|L) = \frac{P(L|A)P(A)}{P(L)}$$

Or in this case

$$P(A|L) = 1 - P(B|L)$$

Bayes rule: lemon cars

Given the above information, what is the probability that it came from factory A?

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$$P(A|L) = \frac{P(L|A)P(A)}{P(L)}$$

Or in this case

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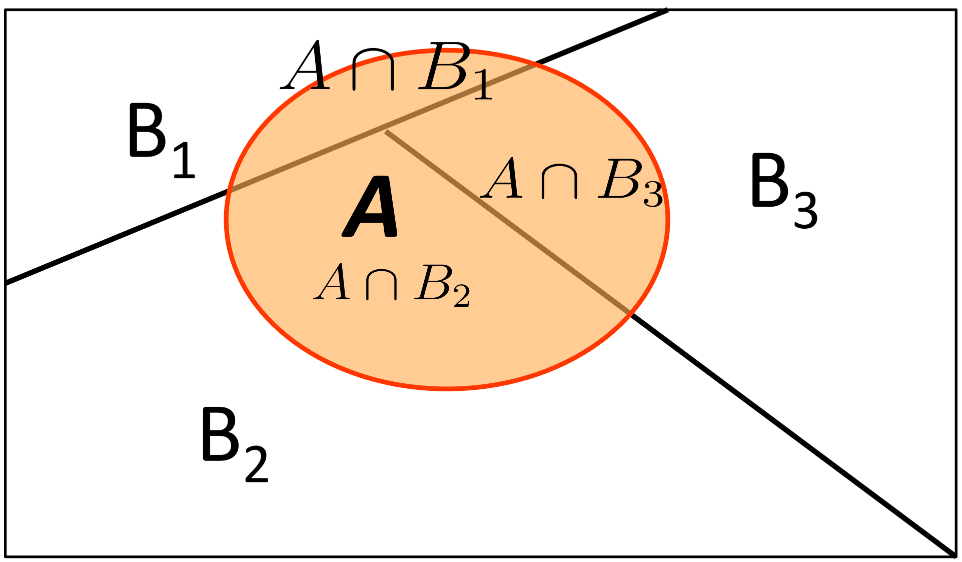
=

Total probability

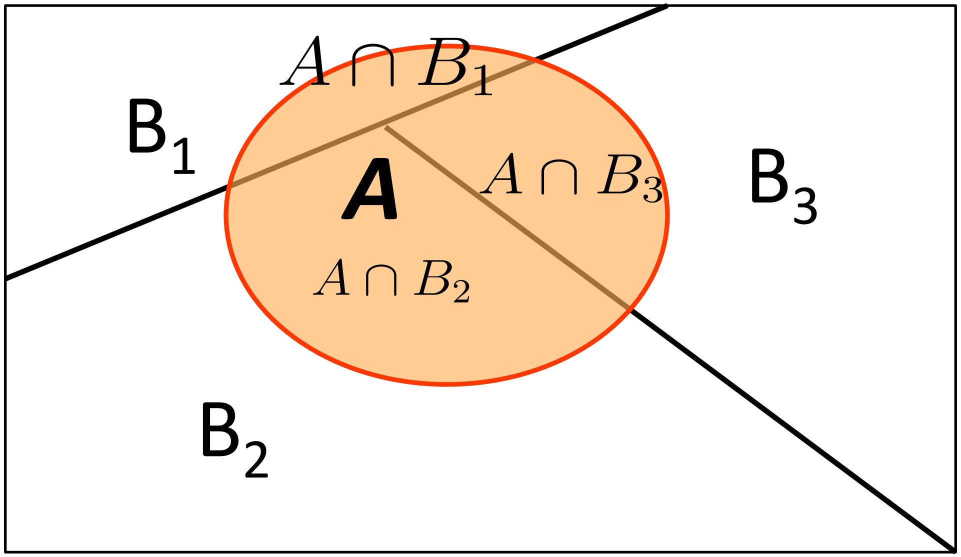
$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$



Total probability general form

$$P(A) = \sum_j (P(A|B_j)P(B_j))$$

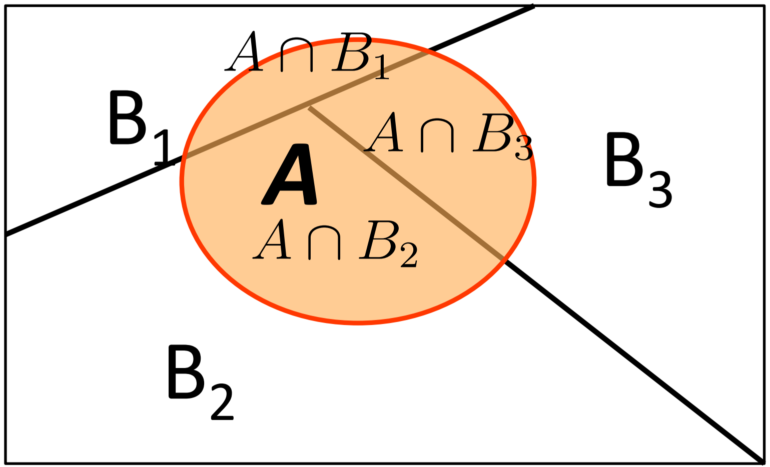
$$if\ B_i \cap B_j = \emptyset\ for\ all\ i \neq j$$



Bayes rule using total prob.

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A)}$$

$$= \frac{P(A|B_j)P(B_j)}{\sum_j P(A|B_j)P(B_j)}$$



Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

Using total prob.

$$= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

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Independence

- One definition:

$$P(A|B) = P(A) \ or$$
$$P(B|A) = P(B)$$

Whether A happened doesn't change the probability of B and vice versa

Independence: example

- Suppose that we have a fair coin and it is tossed twice. let A be the event “the first toss is a head” and B the event “the two outcomes are the same.”

- These two events are

Independence

- Alternative definition

$$P(A|B) = P(A)$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$$

Testing Independence:

- Suppose you draw one card from a standard deck of cards. E_1 is the event that the card is a King, Queen or Jack. E_2 is the event the card is a Heart. Are E_1 and E_2 independent?

Simulation of Conditional Probability

http://www.randomservices.org/random/apps/ConditionalProbabilityExperiment.html

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

Addition material on Counting

Addition principle

- Suppose there are ***n*** disjoint events, the number of outcomes for the union of these events will be the sum of the outcomes of these events.

Multiplication principle

- Suppose that a choice is made in two consecutive stages
  - Stage 1 has ***m*** choices
  - Stage 2 has ***n*** choices
- Then the total number of choices is ***mn***

Multiplication: example

- How many ways are there to draw two cards of the same suit from a standard deck of 52 cards? The draw is without replacement.

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52×12

Permutations (order matters)

- From 10 digits (0,…9) pick 3 numbers for a CS course number (no repetition), how many possible numbers are there?

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10×9×8 = P(10,3) = 720

$$P(n,r) = \frac{n!}{(n-r)!}$$

Combinations (order not important)

- A graph has N vertices, how many edges could there exist at most? Edges are un-directional.

$$C(n,r) = \frac{n!}{(n-r)!r!} = \frac{P(n,r)}{r!} = C(n, n-r)$$

Combinations (order not important)

- A graph has N vertices, how many edges could there exist at most? Edges are un-directional.

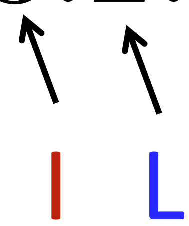
C(N,2) = N×(N-1)/2

$$C(n,r) = \frac{n!}{(n-r)!r!} = \frac{P(n,r)}{r!} = C(n,n-r)$$

Partition

- How many ways are there to rearrange ILLINOIS?

$$\frac{8!}{3!2!1!1!1!}$$



- General form

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

Allocation

- Putting 6 identical letters into 3 mailboxs (empty allowed)

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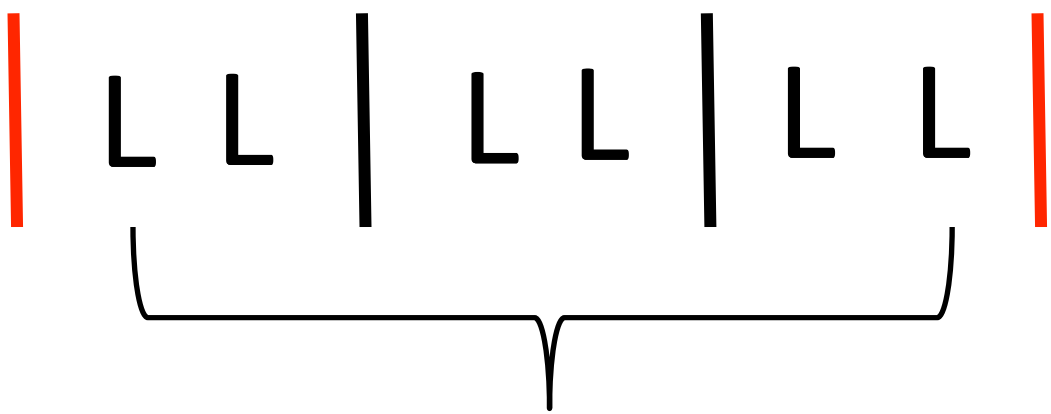

Choose 2 from the 8 positions: **C(8,2) = 28**

Counting: How many think pairs could there be?

- Q. Estimate for # of pairs from different groups. There are **4** even sized groups in a class of 200

Random experiment

- Q: Is the following experiment a random experiment for probabilistic study?

$$2H_{2(g)} + O_{2(g)} \leftrightarrows 2H_2O_{(l)}$$

A. Yes

B. No

Size of sample space

- Q: What is the size of the sample space of this experiment? Deal 5 different cards out of a fairly shuffled deck of standard poker (order matters).

A. C(52,5) **B**. P(52,5) **C**. 52

Event

- Roll a 4-sided die twice

  The event “max is 4” and “sum is 4” are disjoint.

  A. True

  B. False

Probability

- Q: A deck of ordinary cards is shuffled and 13 cards are dealt. What is the probability that the last card dealt is an ace?

  A. 4*P(51,12)/P(52,13) **B**. 4/13

  C. 4*C(51,12)/C(52,13)

Allocation: beads

- Putting 3000 beads randomly into 20 bins (empty allowed)

$$C(3019, 19) = \frac{3019!}{19!3000!}$$

See you next time

See You!