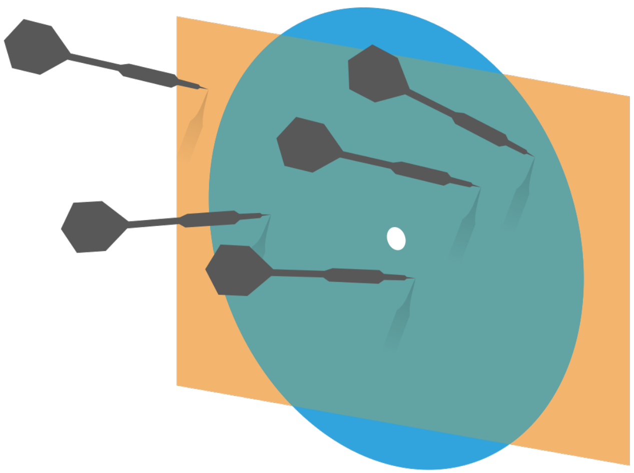
Probability and Statistics for Computer Science



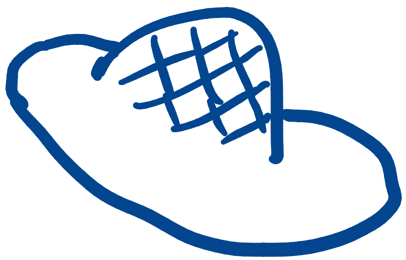
Credit: wikipedia

"A major use of probability in statistical inference is the updating of probabilities when certain events are observed" – Prof. M.H. DeGroot

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 2.11.2021

Counting: how many ways?

if we put 7 hats (indistinguishable) on 7 people out of 10 people randomly?



$$\binom{10}{7}$$

Warm up: which is larger?

$P(A \cap B)$ or $P(A \mid B)$

A) $P(A \cap B)$

$P(B) < 1$

$\boxed{\text{B)}}$ $P(A \mid B)$

C) None

Last time

* More probability calculation using counting

* Conditional probability

  Mutiplication rule;

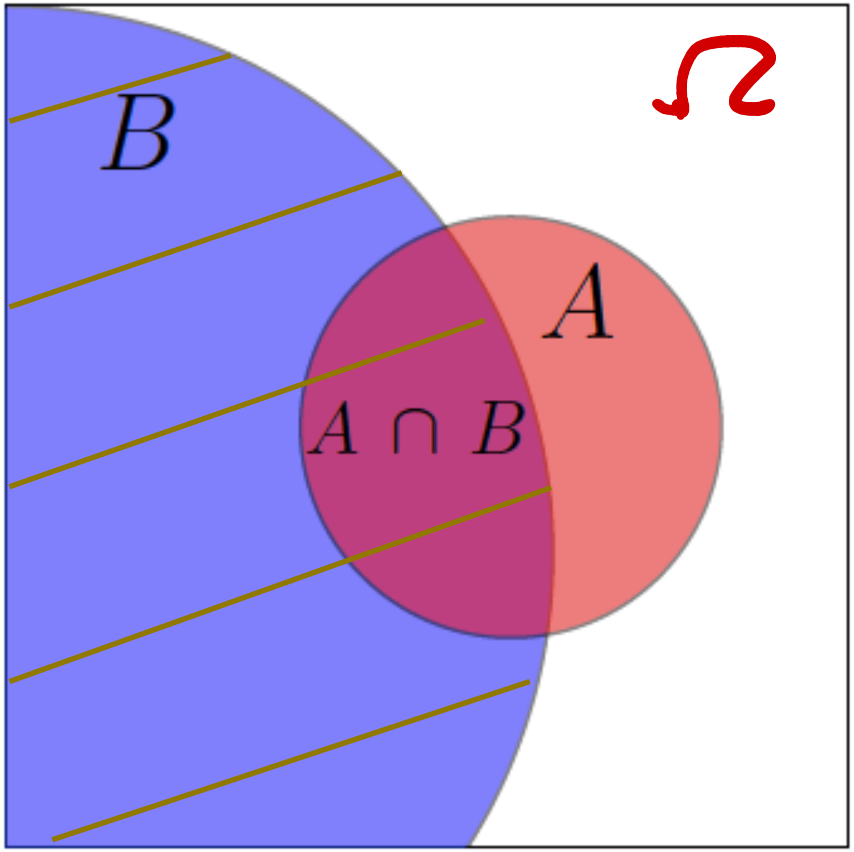
  Bayes rule

Objectives

- Conditional Probability
  - Review
  - Total probability
  - Independence

Conditional Probability

- The probability of ***A*** given ***B***



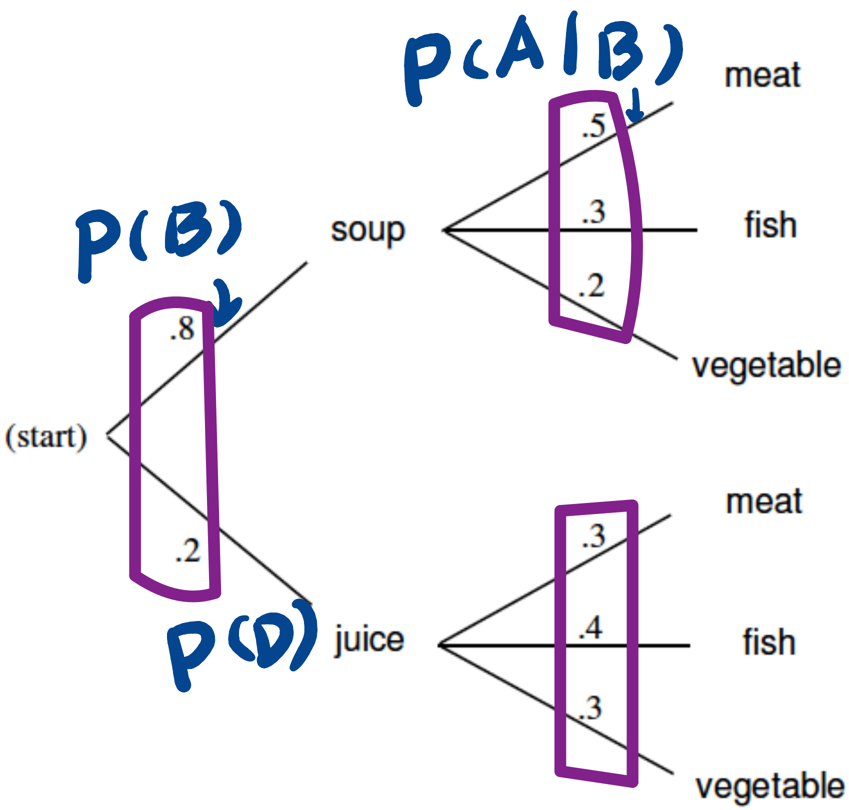
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The line-crossed area is the new sample space for conditional P(A| B)

Joint Probability Calculation

$$\Rightarrow P(A \cap B) = P(A|B)\underline{P(B)} \text{ prior}$$



$P(A^c | B) = ?$ $1 - P(A|B)$ 0.5

$P(A^c \cap D) = ?$

$= P(\text{fish} \cap D) + P(\text{veg} \cap D)$

$= 0.14$

A: meat B: soup

D: juice

$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Joint Probability Calculation

$$\Rightarrow P(A \cap B) = P(A|B)P(B)$$

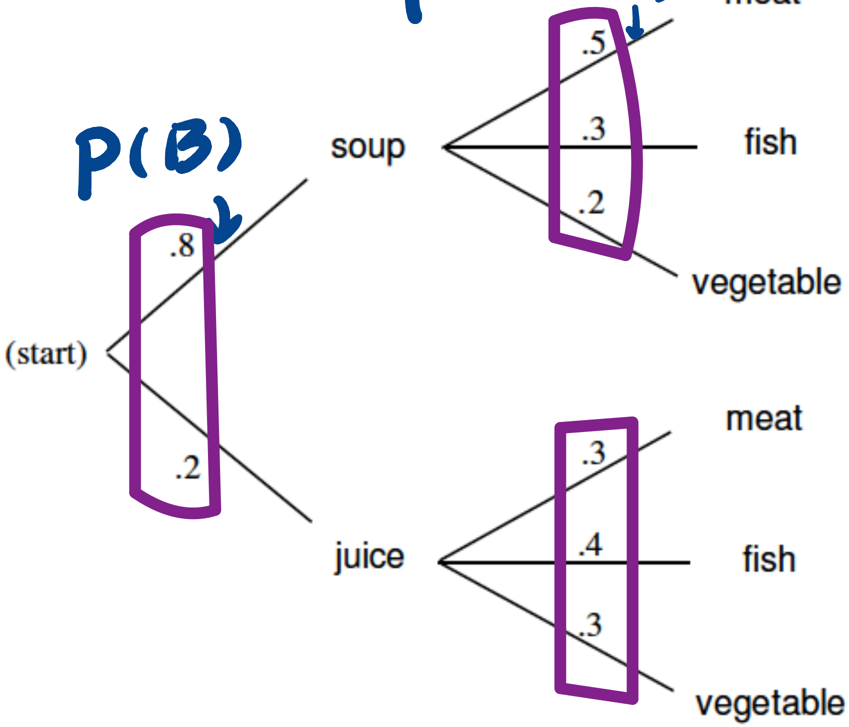
0.6 icecream

0.4 cheesecake

C: cheesecake

$P(A|B)$

$P(B)$



$P(C|B) = ?\ 0.2$

$= \dfrac{P(C \cap B)}{P(B)}$

$P(B \cap A^c \cap C)$

$P(C \cap B)$

$= P(C \cap B \cap A) + \cancel{P(C \cap B \cap A^c)}$

0

$$P(soup \cap meat) =$$
$$P(meat|soup)P(soup)$$
$$= 0.5 \times 0.8 = 0.4$$

Bayes rule

- Given the definition of conditional probability and the symmetry of joint probability, we have:

$$P(A|B)P(B) = P(A \cap B) = P(B \cap A) = P(B|A)P(A)$$

And it leads to the famous Bayes rule:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

Total probability:

$$P(B) = P(A \cap B) + P(A^c \cap B)$$
$$= P(B|A)P(A) + P(B|A^c) \cdot P(A^c)$$

$0.02 \times 0.6 + 0.04 \times 0.4$

$A \to$ rainy

$A^c \to$ sunny

$P(B|A) = \frac{2}{50}$

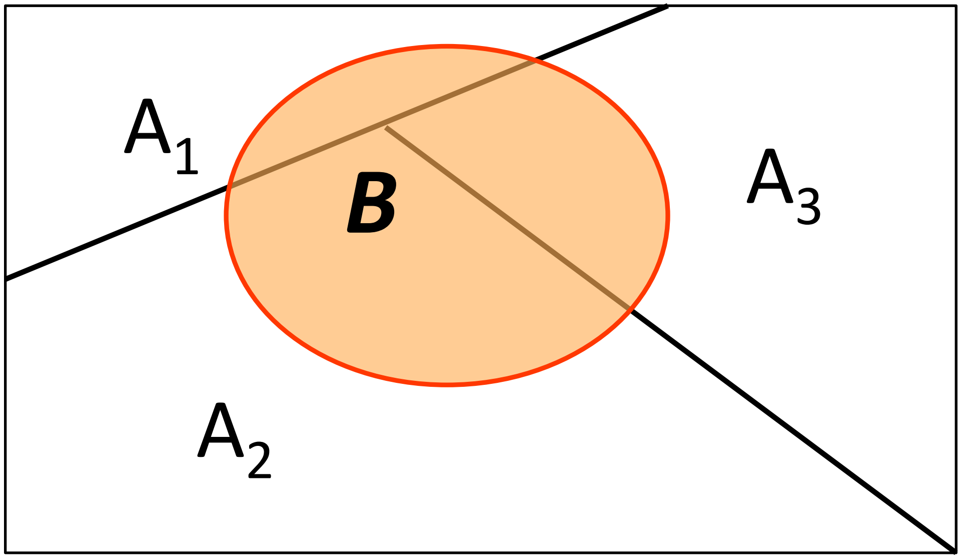
$P(B|A^c) = \frac{2}{100}$

$P(A) = 0.4$

$P(A^c) = 0.6$

Total probability

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$
$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$



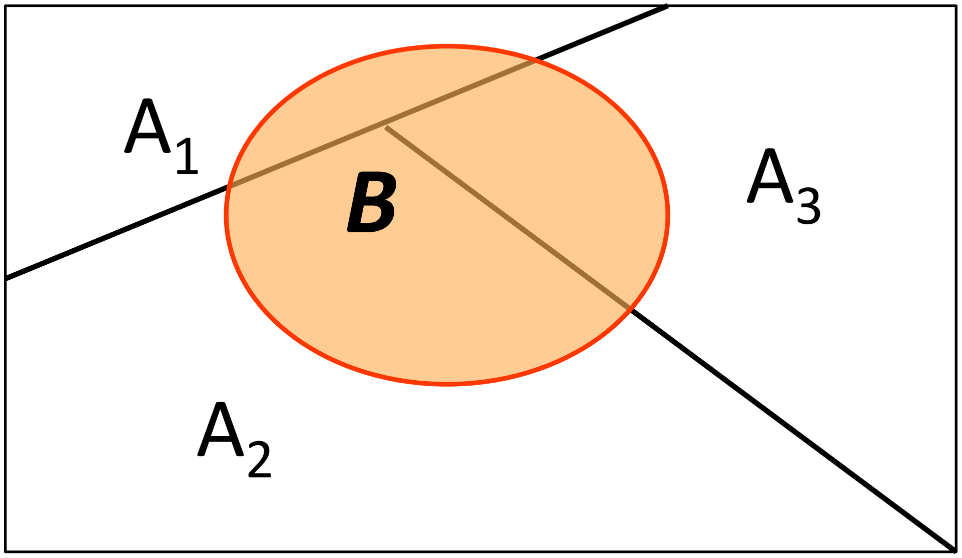
A_1, A_2, A_3 are disjoint,

$A_1 \cup A_2 \cup A_3 \supseteq B$

Total probability general form

$$P(B) = \sum_j P(B \cap A_j)$$

$$= \sum_j P(B \mid A_j) P(A_j)$$

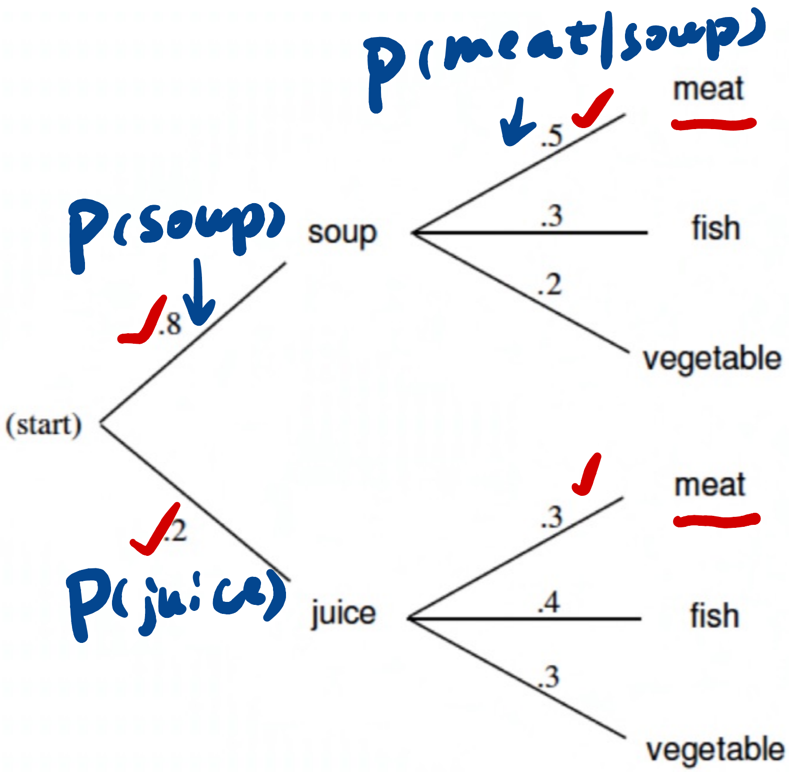


A_j, A_k are disjoint

$A_j \cap A_k = \emptyset$

$j \neq k$

Total probability:



$P(\text{meat}) = ?$

$0.8 \times 0.5 + 0.2 \times 0.3$

$=$

$P(\text{soup} \mid \text{meat}) = ?$

$= \dfrac{P(\text{soup} \cap \text{meat})}{P(\text{meat})}$

$= \dfrac{0.8 \times 0.5}{0.8 \times 0.5 + 0.2 \times 0.3}$

Bayes rule using total prob.

$$P(A_j | B) = \frac{P(B|A_j)\,P(A_j)}{P(B)}$$

$$= \frac{P(B|A_j)\,P(A_j)}{\sum_j P(B|A_j)\,P(A_j)}$$

$A_i \cap A_j = \phi$ → disjoint

if $i \neq j$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is 1/100,000. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability 0.001. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

Using total prob.

$$= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

Bayes rule: rare disease test

There is a blood test for a rare disease. The frequency of the disease is **1/100,000**. If one has it, the test confirms it with probability 0.95. If one doesn't have, the test gives false positive with probability **0.001**. What is $P(D|T)$, the probability of having disease given a positive test result?

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

$= \quad < 1\%$

$P(D) = \frac{1}{100{,}000}$

$P(T|D) = 95\%$

$P(T|D^c) = 0.001$

$P(D^c) = 1 - P(D)$

What about Covid test?

Suppose freq. of Covid = 1.2%

test accuracy = 95%

false positive = 0.001

$P(D|T) = ?$ $\sim 92\%$

Independence

- One definition:

$$P(A|B) = P(A) \;\; or$$
$$P(B|A) = P(B)$$

Whether A happened doesn't change the probability of B and vice versa

Independence: example

- Suppose that we have a fair coin and it is tossed twice. let A be the event "the first toss is a head" and B the event "the two outcomes are the same."

A : H* B : HH TT

$P(A|B) \stackrel{?}{=} P(A)$

$P(A) = \frac{1}{2}$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HH\})}{P(\{HH, TT\})} = \frac{1}{2}$

$P(A|B) = P(A)$

- These two events are independent!

Independence

- Alternative definition

$$P(A|B) = P(A)$$

LHS by definition

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A \cap B) = P(A)P(B)}$$

Testing Independence:

- Suppose you draw one card from a (52) standard deck of cards. E_1 is the event that the card is a King, Queen or Jack. E_2 is the event the card is a Heart. Are E_1 and E_2 independent?

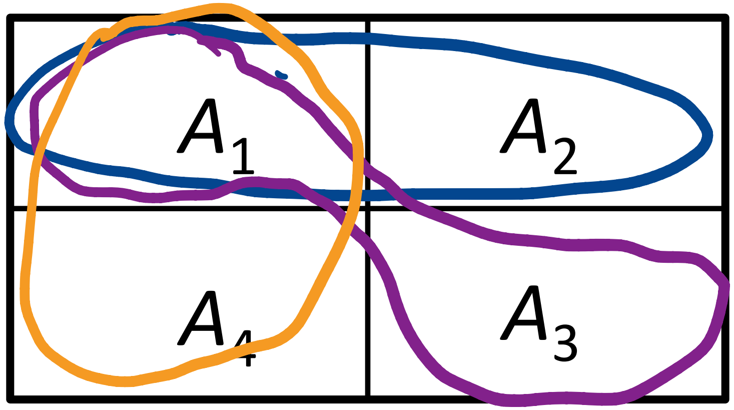
Yes, indpt!

$$P(E_1) = \frac{3\times 4}{52} = \frac{3}{13}$$

$$P(E_2) = \frac{1}{4}$$

$$P(E_1 \cap E_2) = \frac{3}{52} = P(E_1)P(E_2) = \frac{3}{52}$$

Pairwise independence is not mutual independence in larger context



$A_1 \cup A_2 \cup A_3 \cup A_4 = \Omega$

$P(A_1) = P(A_2) = P(A_3) = P(A_4) = 1/4$

$A = A_1 \cup A_2; P(A) = \frac{1}{2}$

$B = A_1 \cup A_3; P(B) = \frac{1}{2}$

$C = A_1 \cup A_4; P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4} = P(A)P(B) = \frac{1}{4}$ ✓

$P(A \cap C) = \frac{1}{4} = P(A)P(C) = \frac{1}{4}$ ✓

$P(B \cap C) = \frac{1}{4} = P(B)P(C) = \frac{1}{4}$ ✓

$P(A \cap B \cap C) = P(A_1) = \frac{1}{4}$

$\neq P(A)P(B)P(C) = \frac{1}{8}$

$*P(ABC)$ *is the shorthand for* $P(A \cap B \cap C)$

Mutual independence

- Mutual independence of a collection of events $A_1, A_2, A_3 \ldots A_n$ is :

$$P(A_i | A_j \cap A_k \cap \ldots \cap A_p) = P(A_i)$$

$$j, k, \ldots p \neq i$$

- It's very strong independence!

$P(A_1 \cap A_2 \cdots \cap A_p) = P(A_1) P(A_2) \cdots P(A_p)$

for any p # of events.

Probability using the property of Independence: Airline overbooking (1)

- An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P (overbooked)

E : 7 showed up

Indpt Assump.

P (A_1 showed $\cap$ A_2 showed . . . . . A_7)

= P(A_1)P(A_2) . . . P(A_7)

Probability using the property of Independence: Airline overbooking (1)

- An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P(7 passengers showed up) $= \overbrace{p \cdot p \cdot p \quad p}^{7}$

$= p^7$

Probability using the property of Independence: Airline overbooking (2)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that exactly 6 people showed up?

1 2 3 4 5 6

1 3 4 6 7 8

P(6 people showed up) = $\binom{8}{6} \cdot p^6 (1-p)^2$

Probability using the property of Independence: Airline overbooking (3)

- An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P(overbooked) = $\sum_{u=7}^{8} \binom{8}{u} p^u (1-p)^{8-u}$

Probability using the property of Independence: Airline overbooking (4)

- An airline has a flight with **s** seats. They always sell **t** (**t**>**s**) tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that exactly **u** people showed up?

P(exactly **u** people showed up)

$$= \binom{t}{u} p^{u} (1-p)^{t-u}$$

Probability using the property of Independence: Airline overbooking (5)

- An airline has a flight with **s** seats. They always sell **t** (**t**>**s**) tickets for this flight. If ticket holders show up independently with probability **p**, what is the probability that the flight is overbooked ?

P(overbooked) $\sum_{u=s+1}^{t} \binom{t}{u} \cdot p^{u} (1-p)^{t-u}$

Independence vs Disjoint

- Q. Two disjoint events that have probability> 0 are certainly dependent to each other.

  $\boxed{\text{A.}}$ True

  B. False

$$P(A\cap B) \overset{?}{\neq} P(A)P(B)$$

$$P(A) > 0$$

$$P(B) > 0$$

$$P(\overset{=\emptyset}{A\cap B}) = 0$$

6-sided

$\{1, 2, 3, 4, 5, 6\} \to \Omega$

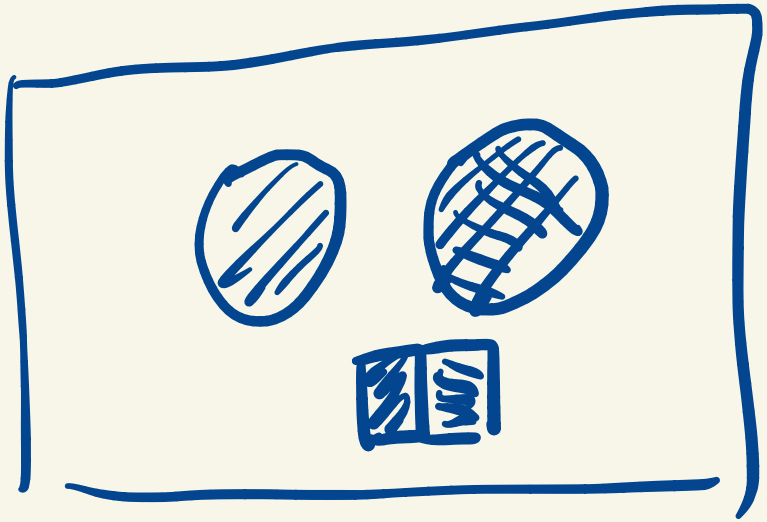
$E_1 = \{1, 2, 4\}$ 2

$E_2 = \{3, 5\}$

E_3: even

E_1 & E_2 are disjoint

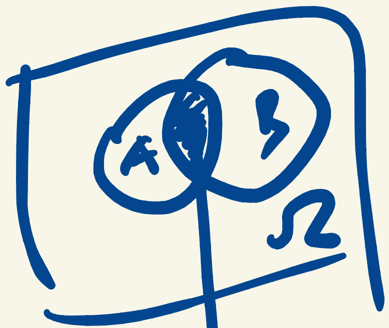
$E_1 \cap E_2 = \phi$



indpt

$P(A|B) = P(A)$

$P(B|A) = P(B)$



$C = A \cap B$

$P(C) = P(A)P(B)$

indpt. $P(A \cap B) = P(A)P(B)$

Assignments

- Module week3 on Compass
- Next time: Random variable

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!