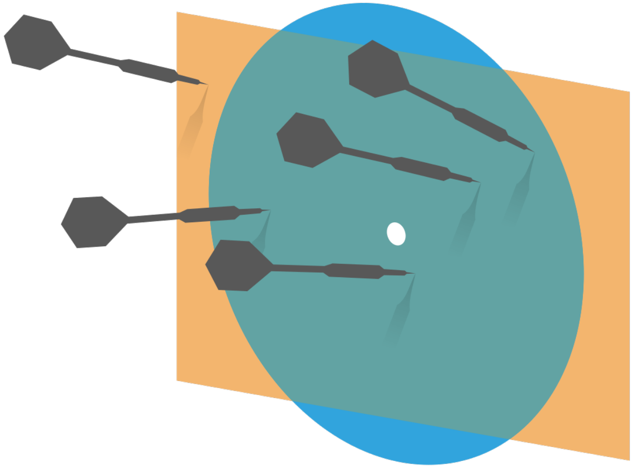
Probability and Statistics for Computer Science



Credit: wikipedia

"It's straightforward to link a number to the outcome of an experiment. The result is a **Random variable**." ---Prof. Forsythe

Random variable is a function, it is not the same as in **X = X+1**

Random numbers

- Amount of money on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- Body temperature of a puppy in its pet clinic
- Level of the intensity of pain in a toothache

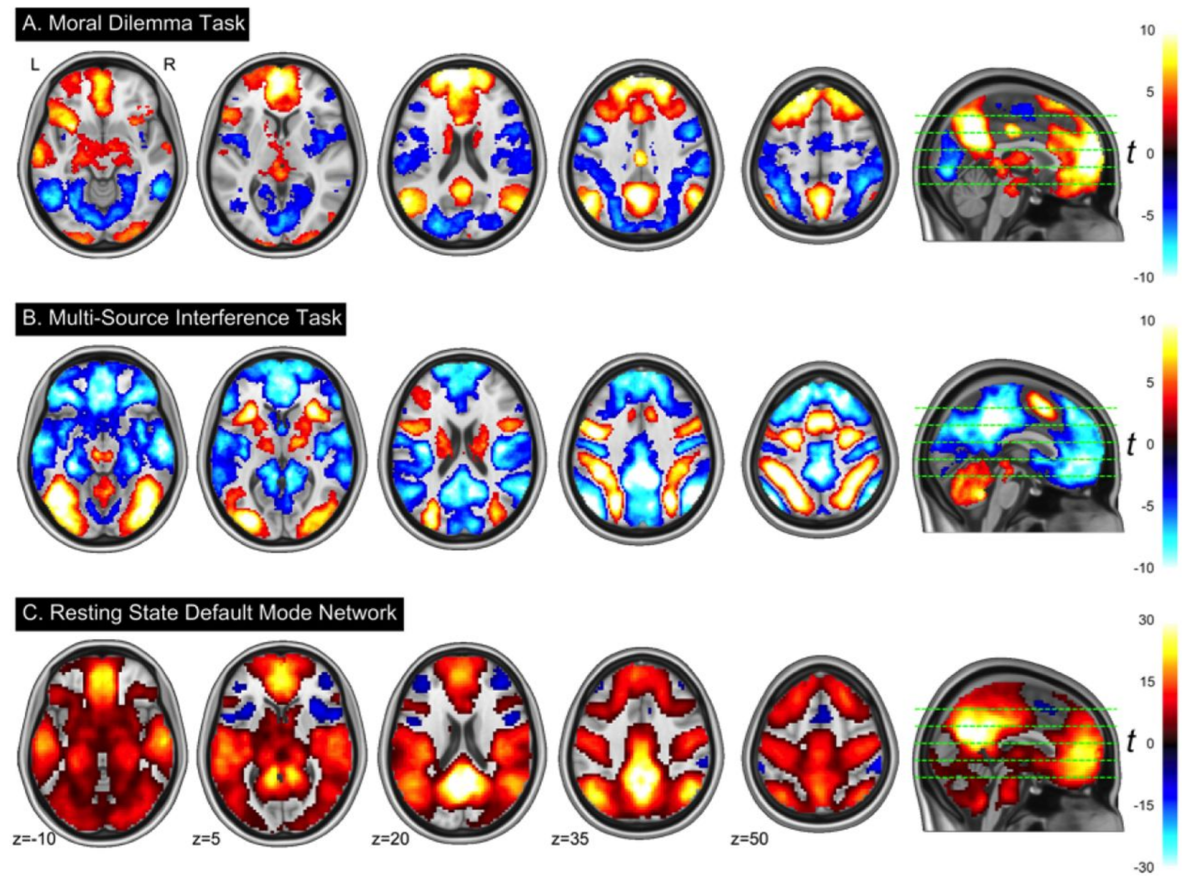
Random variable as vectors

Brain imaging of Human emotions

A) Moral conflict

B) Multi-task

C) Rest

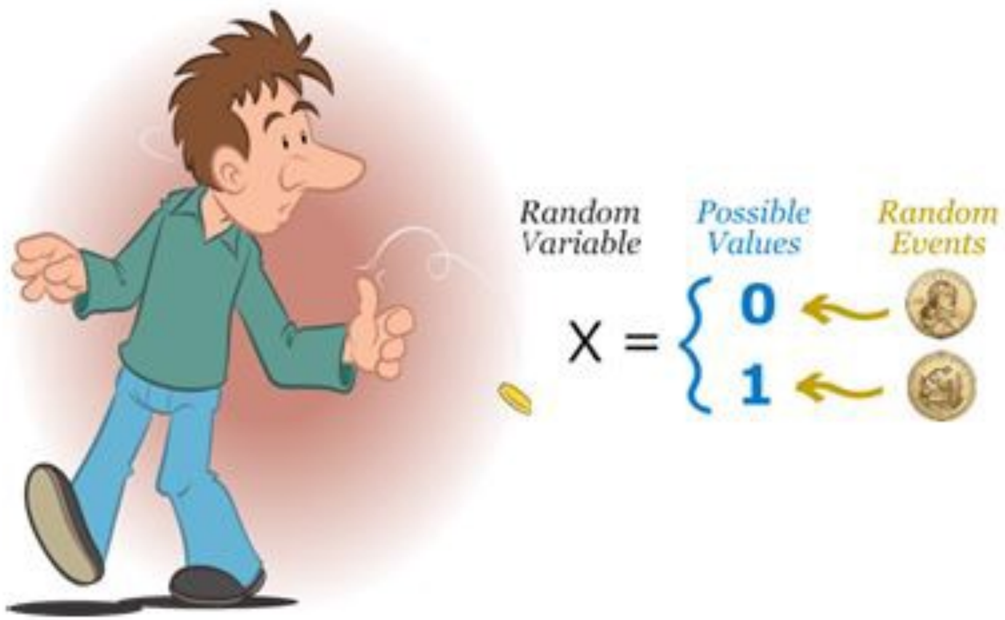


A. McDonald et al. NeuroImage doi: 10.1016/j.neuroimage.2016.10.048

Content

- Random Variable
- Probability distribution
- Cumulative distribution
- Joint probability
- Independence of random variables

Random variables



Random variables

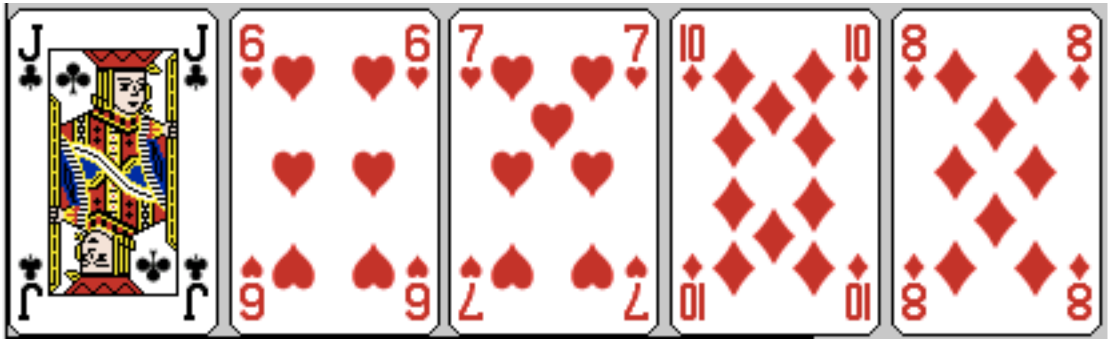
- The values of a random variable can be either **discrete**, **continuous** or **mixed**.

Discrete Random variables

- The range of a discrete random variable is a countable set of real numbers.

Random Variable Example

- **Number of pairs in a hand of 5 cards**



- Let a single outcome be the hand of 5 cards
- Each outcome maps to values in the set of numbers {0, 1, 2}

Random Variable Example

- **Number of pairs in a hand of 6 cards**
  - Let a single outcome be the hand of 6 cards
  - What is the range of values of this random variable?

Q: Random Variable

- If we roll a 3-sided fair die, and define random variable U, *such that*

A. {-1, 0, 1}

B. {0, 1}

Three important facts of Random variables

- Random variables have **probability functions**
- Random variables can be **conditioned** on events or other random variables
- Random variables have **averages**

Random variables have **probability functions**

- Let X be a random variable
- The set of outcomes

  is an event with probability

$$P(X = x_0)$$

X is the random variable

x_0 is any unique instance that X takes on

Probability Distribution

- $P(X = x)$ is called the probability distribution for all possible $\boldsymbol{x}$
- $P(X = x)$ is also denoted as $P(x)$ or $p(x)$
- $P(X = x) \geq 0$ for all values that X can take, and is 0 everywhere else
- The sum of the probability distribution is 1 $$\sum_x P(x) = 1$$

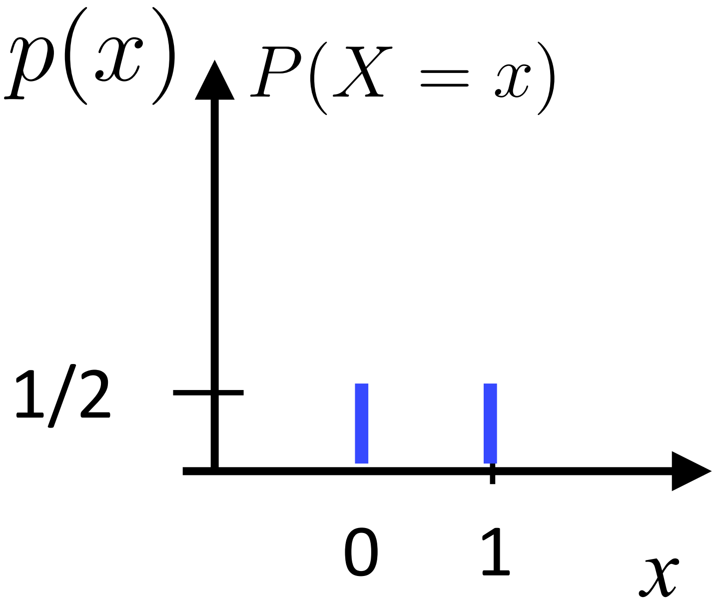
Cumulative distribution

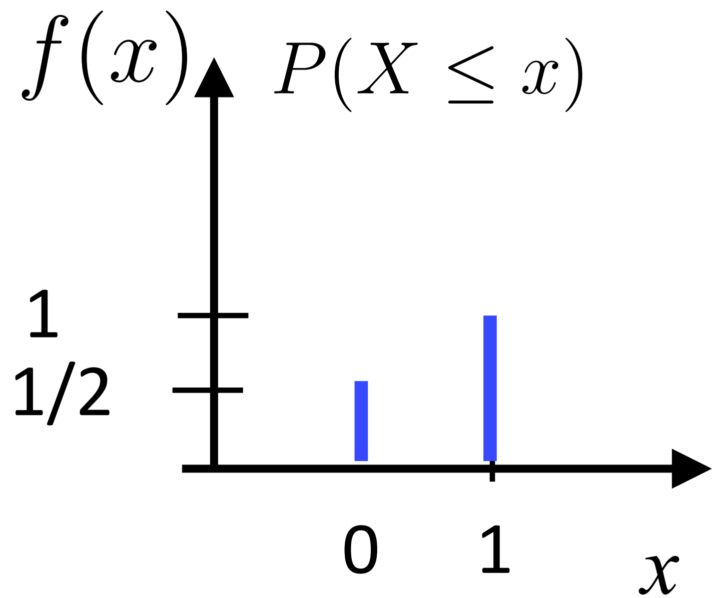
- $P(X \leq x)$ is called the cumulative distribution function of X
- $P(X \leq x)$ is also denoted as $f(x)$
- $P(X \leq x)$ is a non-decreasing function of x

Probability distribution and cumulative distribution

- Give the random variable ***X***,

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$





Function of random variables: die example

Roll 4-sided fair die twice.

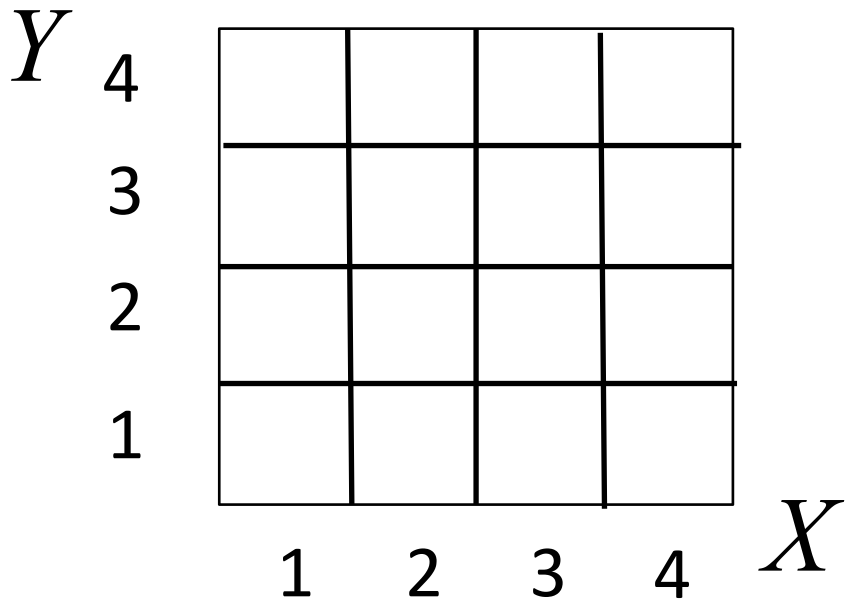
Define these random variables:

X, the values of 1st roll

Y, the values of 2nd roll

Sum $S = X + Y$

Difference $D = X - Y$



Size of Sample Space = ?

Random variable: die example

$S = X + Y$

Y					
4	5	6	7	8	
3	4	5	6	7	
2	3	4	5	6	
1	2	3	4	5	
	1	2	3	4	X

$$P(S = 7)$$

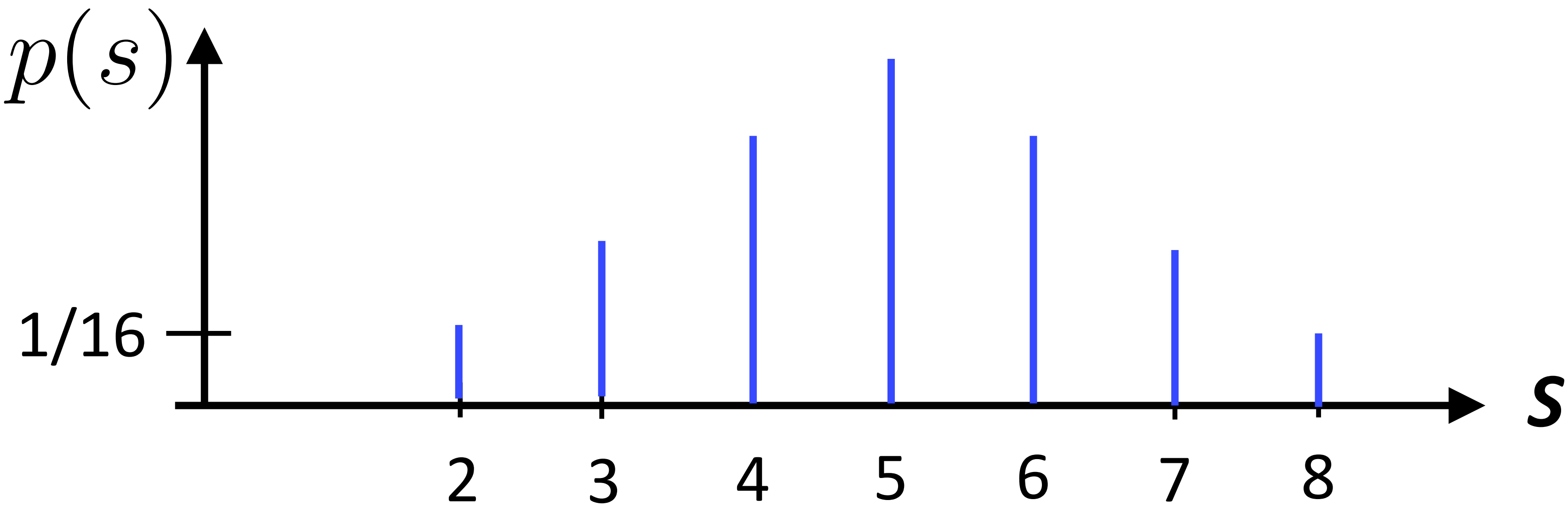
$D = X - Y$

Y					
4	-3	-2	-1	0	
3	-2	-1	0	1	
2	-1	0	1	2	
1	0	1	2	3	
	1	2	3	4	X

$$P(D \leq -1)$$

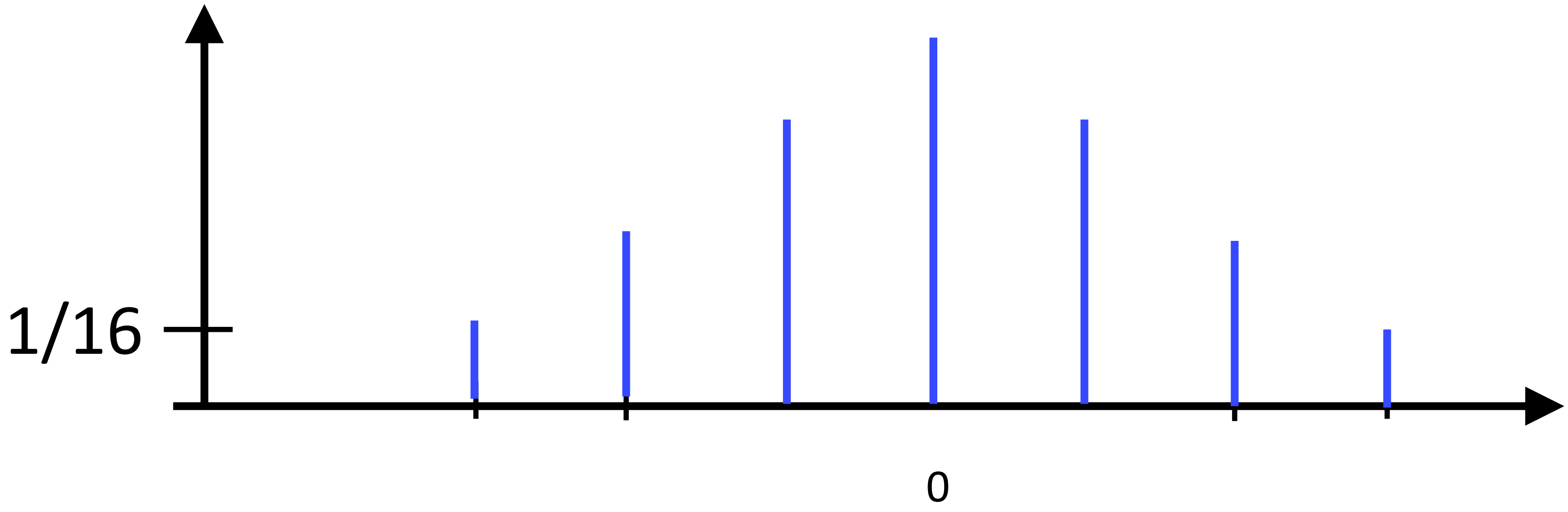
Probability distribution of the sum of two random variables

- Give the random variable ***S*** *in the 4-sided die, whose range is* {2,3,4,5,6,7,8}, probability distribution of ***S*.**



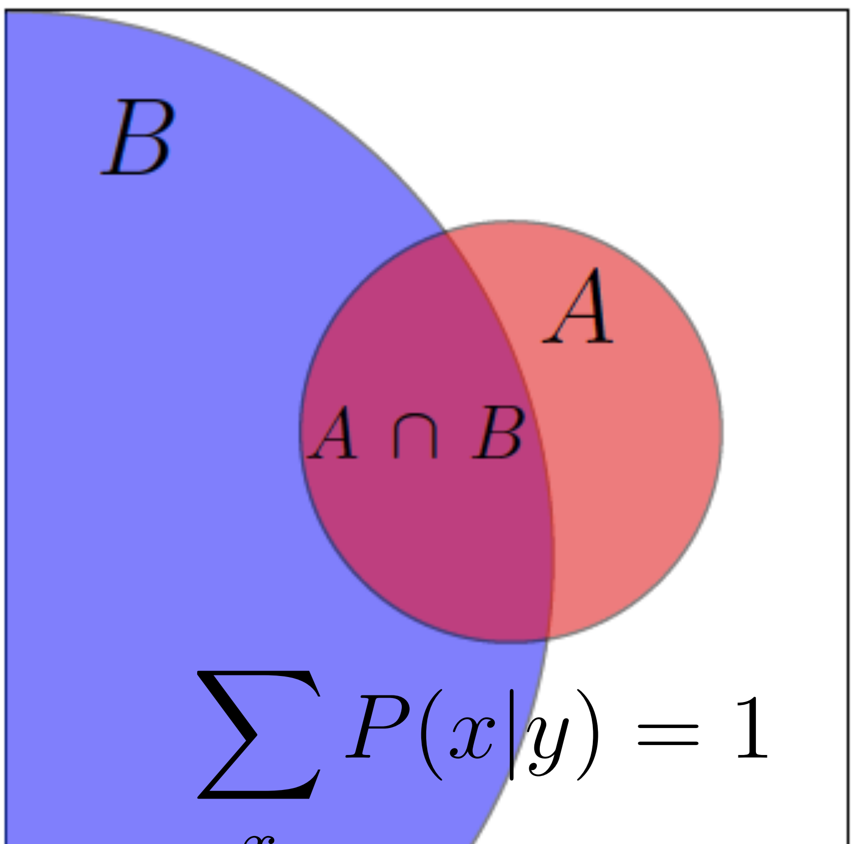
Probability distribution of the difference of two random variables

- Give the random variable ***D = X-Y****, what is the probability distribution of* ***D****?*



Conditional Probability

- The probability of ***A*** given ***B***



$$\sum_{x} P(x|y) = 1$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The “Size” analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$$P(x|y) = \frac{P(x,y)}{P(y)} \qquad P(y) \neq 0$$

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad P(y) \neq 0$$

- The joint probability distribution of two random variables ***X*** and ***Y*** is

$$P(\{X = x\} \cap \{Y = y\})$$

$$\sum_x P(x|y) = 1$$

Get the marginal from joint distri.

- We can recover the individual probability distributions from the joint probability distribution

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

Joint probabilities sum to 1

- The sum of the joint probability distribution

$$\sum_{y}\sum_{x} P(x,y) = 1$$

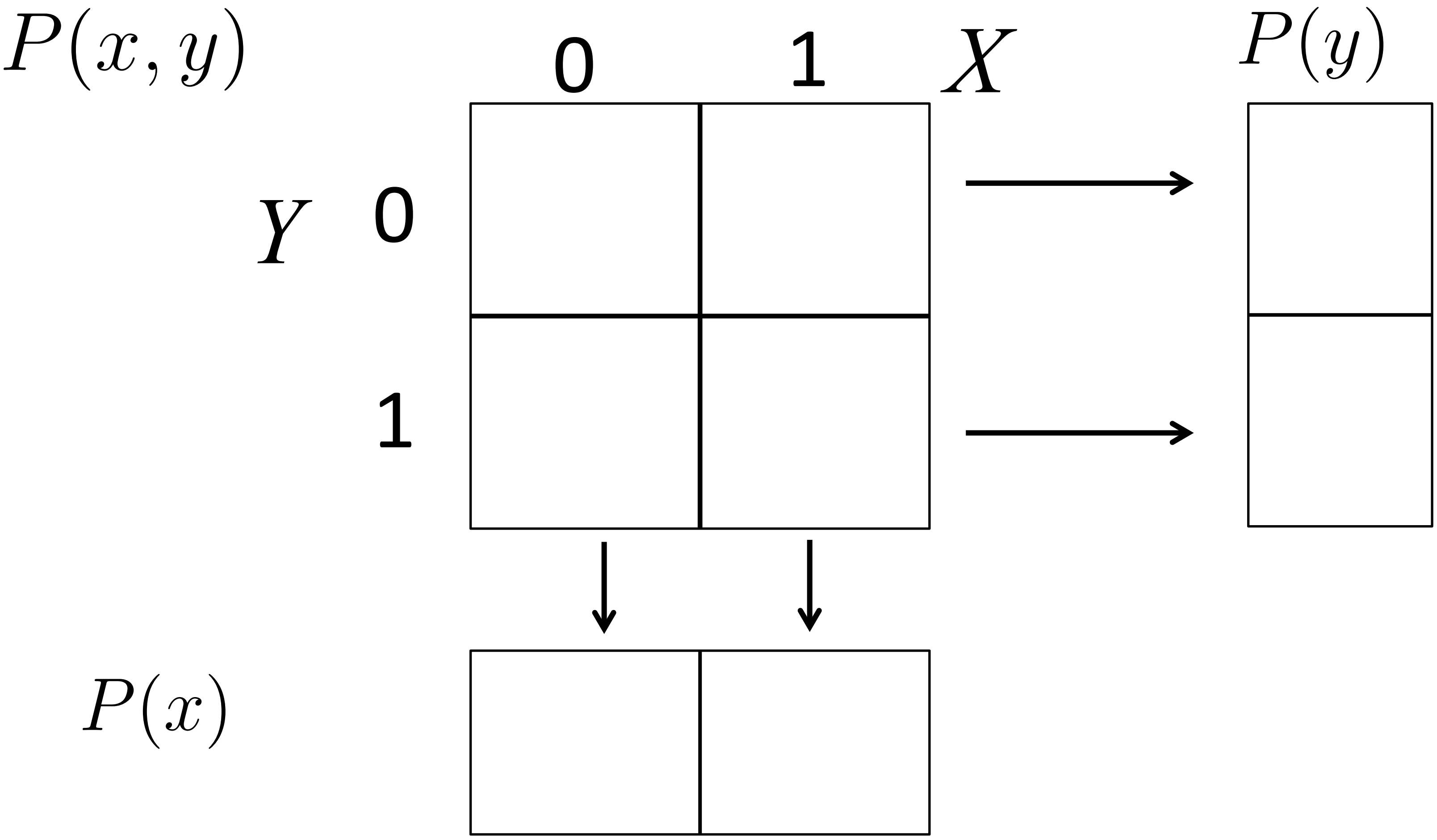
Joint Probability Example

- Tossing a coin twice, we define random variable $X\ and\ Y$ for each toss.

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint probability distribution example



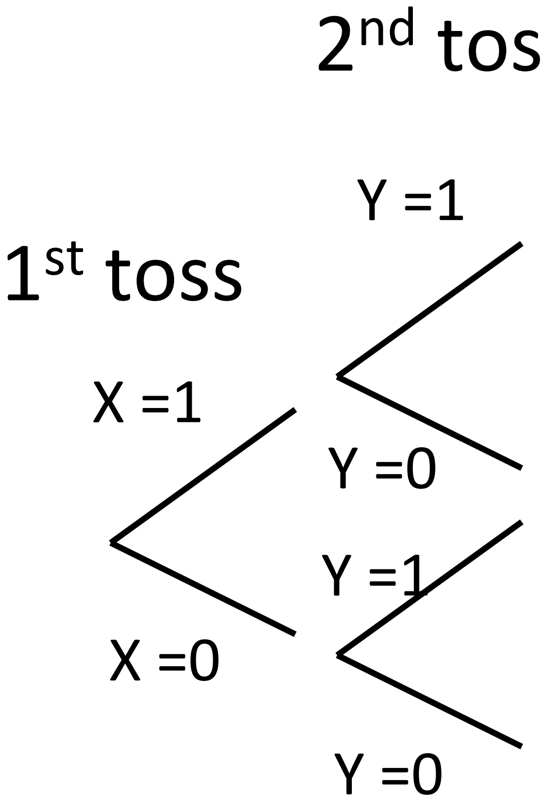
Joint Probability Example

Now we define Sum ***S*** = $X + Y$, Difference ***D*** = $X - Y$. ***S*** takes on values {0,1,2} and ***D*** takes on values {-1, 0, 1}

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

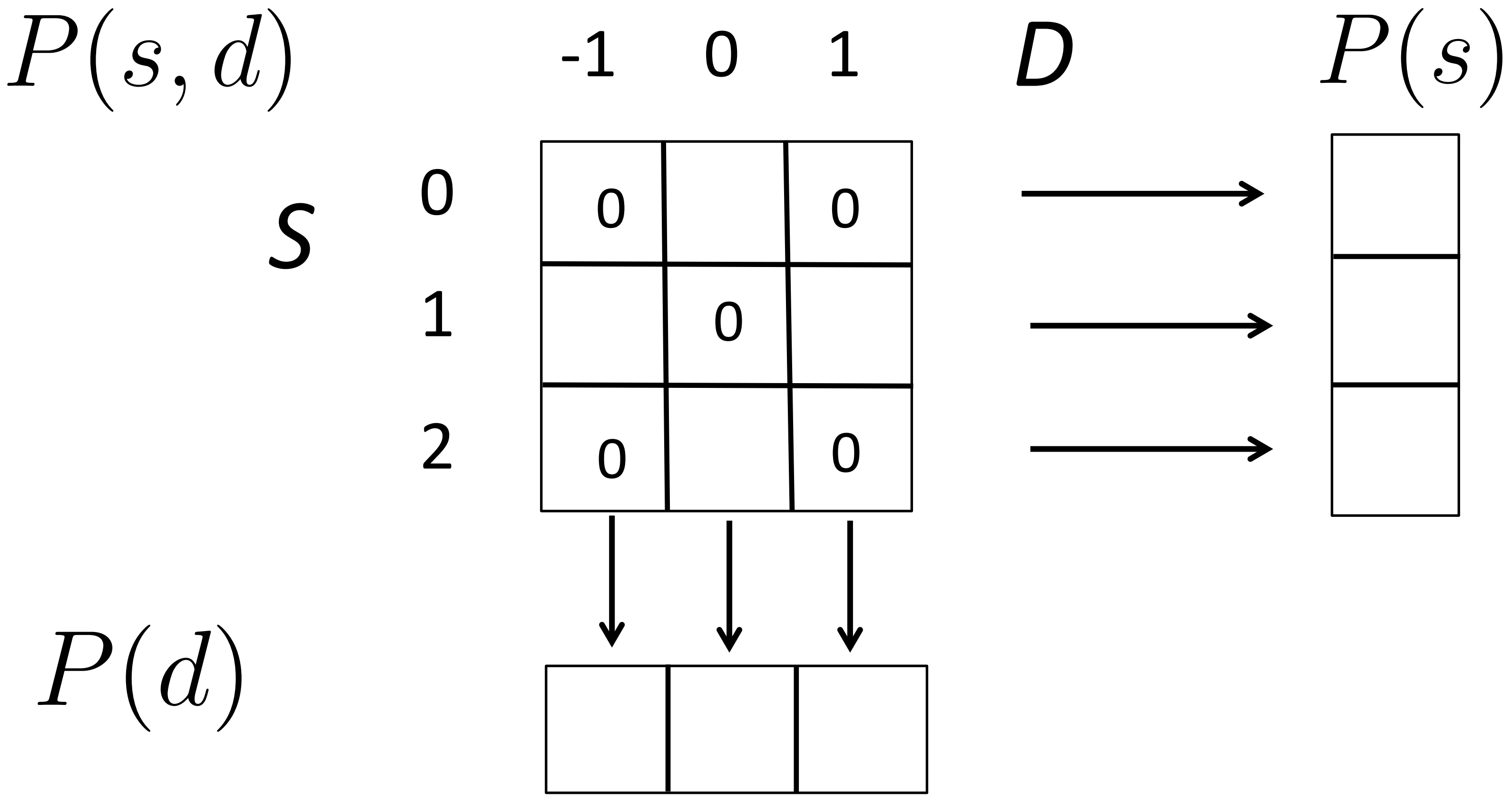
Joint Probability Example



S	D	P(s, d)
2	0	
1	1	
1	-1	
0	0	

Suppose coin is fair, and the tosses are independent

Joint probability distribution example



Independence of random variables

- Random variable $X \, and \, Y$ are independent if

$$P(x,y) = P(x)P(y) \; \boxed{for \; all \; x \; and \; y}$$

- In the previous coin toss example
  - Are X and Y independent?
  - Are ***S*** and ***D*** independent?

Joint probability distribution example

$P(x, y)$

		0	1	X	$P(y)$
Y	0	$\frac{1}{4}$	$\frac{1}{4}$	$\longrightarrow$	$\frac{1}{2}$
	1	$\frac{1}{4}$	$\frac{1}{4}$	$\longrightarrow$	$\frac{1}{2}$
		$\downarrow$	$\downarrow$		
$P(x)$		$\frac{1}{2}$	$\frac{1}{2}$		

Joint probability distribution example

$P(s,d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	$\boxed{0}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

0	$\frac{1}{4}$
1	$\boxed{\frac{1}{2}}$
2	$\frac{1}{4}$

$P(d)$

-1	0	1
$\frac{1}{4}$	$\boxed{\frac{1}{2}}$	$\frac{1}{4}$

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

Bayes rule for random variable

- Bayes rule for events generalizes to random variables

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\boxed{\begin{aligned} P(x|y) &= \frac{P(y|x)P(x)}{P(y)} \\ &= \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \end{aligned}}$$

Total Probability

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

$$P(D=-1|S=1) = \frac{P(S=1|D=-1)P(D=-1)}{P(S=1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}}$$

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability”
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics”

See you next time

See You!