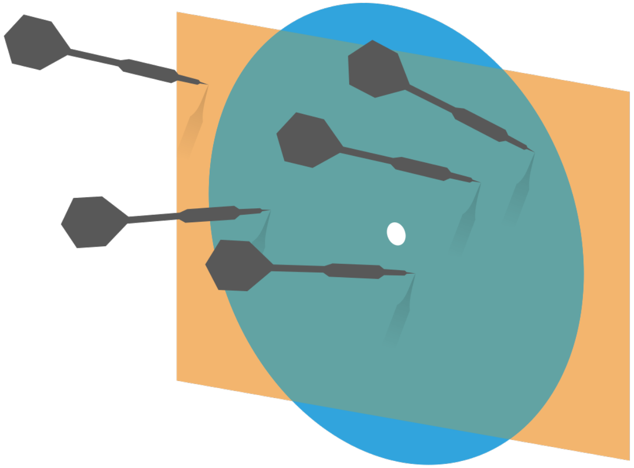
Probability and Statistics for Computer Science



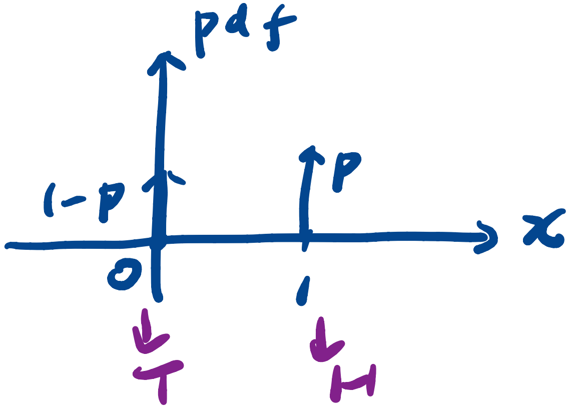
Credit: wikipedia

"I have now used each of the terms mean, variance, covariance and standard deviation in two slightly different ways." ---Prof. Forsythe

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 2.18.2021

Last time

- Random Variable
- Probability distribution
- Cumulative distribution



$P(X = x)$?

$P(X=1)$

Objectives

- Random Variable
  - *Conditional & Joint Probability*
  - *Expected value*
  - *Variance & covariance*

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$P(X=x|Y=y)$

$$P(x|y) = \frac{P(x,y)}{P(y)} \qquad P(y) \neq 0$$

$P(x,y) = P(X=x \cap Y=y)$

$P(y) = P(Y=y)$

$P(x|y) = P(X=x|Y=y)$

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad P(y) \neq 0$$

for all x

- The joint probability distribution of two random variables ***X*** and ***Y*** is

$$P(\{X = x\} \cap \{Y = y\})$$

$$\sum_x P(x|y) = 1$$

Get the marginal from joint distri.

- We can recover the individual probability distributions from the joint probability distribution

$$P(x) = \sum_y P(x,y)$$

RHS $= \sum_y p(y|x)\, p(x)$

$= p(x) \cdot \cancel{\sum_y p(y|x)}$ 1

$$P(y) = \sum_x P(x,y)$$

Joint probabilities sum to 1

- The sum of the joint probability distribution

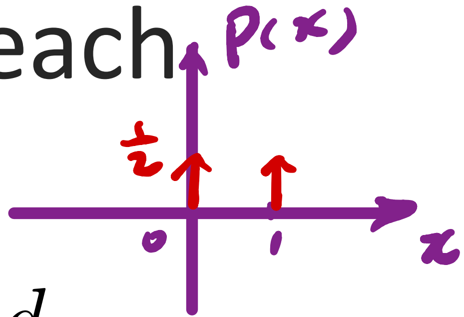
$$\sum_y \sum_x P(x, y) = 1$$

$LHS = \sum_y P(y)$

$= 1$

Joint Probability Example

- Tossing a fair coin twice, we define random variable X *and* Y for each toss.



$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint probability distribution example

$P(x, y)$

$P(X=x \cap Y=y)$

		X = 0	X = 1
Y	0	$p(x=0, y=0)$ $p(0,0)$ $\frac{1}{4}$	$p(x=1, y=0)$ $p(1,0)$ $\frac{1}{4}$
	1	$p(0,1)$ $\frac{1}{4}$	$p(1,1)$ $\frac{1}{4}$

$P(y)$

$\frac{1}{2} = P(y=0)$
$\frac{1}{2} = P(y=1)$

$P(x)$

$\frac{1}{2}$	$\frac{1}{2}$
$p(x=0)$	$p(x=1)$

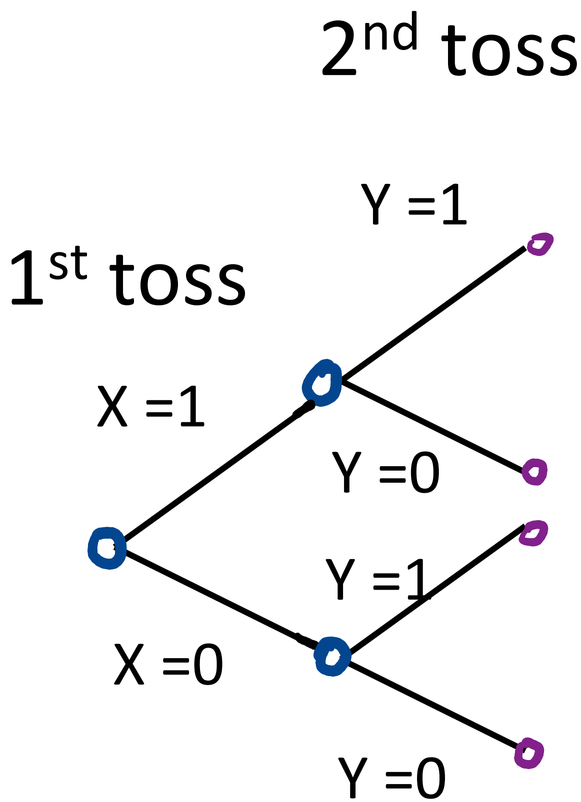
Joint Probability Example

Now we define Sum ***S*** = $X + Y$, Difference ***D*** = $X - Y$. ***S*** takes on values {0,1,2} and ***D*** takes on values {-1, 0, 1}

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint Probability Example



$P(S=s \cap D=d)$

S	D		P(s, d)
2	0	HH	$\frac{1}{4}$
1	1	HT	$\frac{1}{4}$
1	-1	TH	$\frac{1}{4}$
0	0	TT	$\frac{1}{4}$

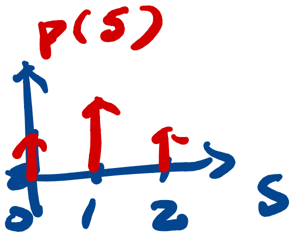
Suppose coin is fair, and the tosses are independent

Joint probability distribution example

$P(s,d)$

S \ D	-1	0	1	$P(s)$
0	0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
2	0	$\frac{1}{4}$	0	$\frac{1}{4}$

$\rightarrow P(S=0)$



$P(d)$

-1	0	1
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(S=s)$

Independence of random variables

- Random variable $X\ and\ Y$ are independent if

$$P(x,y) = P(x)P(y)\ \boxed{for\ all\ x\ and\ y}$$

- In the previous coin toss example
  - Are X and Y independent?
  - Are ***S*** and ***D*** independent?

Joint probability distribution example

$P(x, y)$

	X: 0	X: 1	$P(y)$
Y: 0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
Y: 1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$

$p(x,y) \overset{\checkmark}{=} p(x)p(y)$

Joint probability distribution example

$P(s,d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	$\boxed{0}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

s	$P(s)$
0	$\frac{1}{4}$
1	$\boxed{\frac{1}{2}}$
2	$\frac{1}{4}$

$P(d)$

-1	0	1
$\frac{1}{4}$	$\boxed{\frac{1}{2}}$	$\frac{1}{4}$

$p(s=1 \cap d=0) = 0$

$p(s=1) = \frac{1}{2}$

$p(d=0) = \frac{1}{2}$

$p(s,d) \neq p(s)p(d)$

Joint probability distribution example

$P(s,d)$

S \ D	-1	0	1	$P(s)$
0	0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	$\frac{1}{4}$	$\boxed{0}$	$\frac{1}{4}$	$\boxed{\frac{1}{2}}$
2	0	$\frac{1}{4}$	0	$\frac{1}{4}$

$P(d)$

$\frac{1}{4}$	$\boxed{\frac{1}{2}}$	$\frac{1}{4}$

$$P(S=1, D=0) \neq P(S=1)P(D=0)$$

After class

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	$\boxed{1}$	0	$\boxed{1}$
2	0	$\frac{1}{2}$	0

$$P(S=1 \mid D=-1) = \frac{P(S=1, D=-1)}{P(D=-1)} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

After class

Bayes rule for random variable

- Bayes rule for events generalizes to random variables

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\boxed{\begin{aligned} P(x|y) &= \frac{P(y|x)P(x)}{P(y)} \\ &= \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \end{aligned}}$$

Total Probability

After class

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

$$P(D=-1|S=1) = \frac{P(S=1|D=-1)P(D=-1)}{P(S=1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}}$$

Three important facts of Random variables

- Random variables have **probability functions**
- Random variables can be **conditioned** on events or other random variables
- Random variables have **averages**

Expected value

- The **expected value** (or **expectation**) of a random variable X is

≤ 1

≥ 0

$$E[X] = \sum_x xP(x)$$

$\rightarrow P(X=x)$

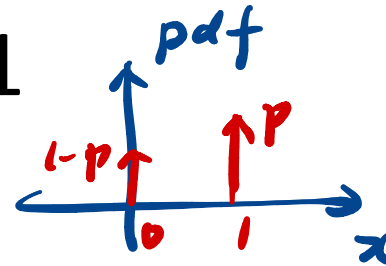
$\sum_x P(x) = 1$

The expected value is a weighted sum of **all** the values X can take

Expected value

- The **expected value** of a random variable X is

<= 1



$$E[X] = \sum_x x \boxed{P(x)}$$

$\sum_x p(x) = 1$

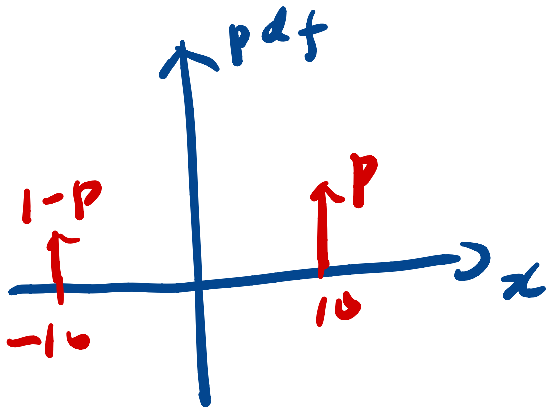
$E[X] = 0.5$ if $p = 1/2$

Theoretical mean

The expected value is a weighted sum of all the values X can take

Expected value: profit

- A company has a project that has ***p*** probability of earning 10 million and ***1-p*** probability of losing 10 million.
- Let X be the return of the project.



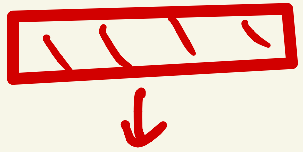
$$E[X] = \sum x\,p(x)$$
$$= (-10)\cdot(1-p) + 10\cdot p$$
$$= 20p - 10 \geq 0$$
$$p \geq \frac{1}{2}$$

After class

Cookie

$1 each

Chocolate



$2 each

A) random draw 1 out of 4

Expected value?

B) random draw 1 twice with replacement

when the two are the same, you get the prize.

Expected value?

Linearity of Expectation

- For random variables X and Y and constants k,c
  - Scaling property
    $$E[kX] = kE[X]$$
  - Additivity
    $$E[X+Y] = E[X] + E[Y]$$
  - And $E[kX+c] = kE[X] + c$

Linearity of Expectation

- Proof of the additive property

$$E[X+Y] = E[X] + E[Y]$$

$S = X + Y$

$$E[X+Y] = E(S) = \sum_s s\, P(s)$$

$$= \sum_{\{s=x+y\}} s \sum_{\{s=x+y\}} P(x,y)$$

$$= \sum_x \sum_y (x+y) P(x,y)$$

$P(s) = P(S=s)$

Note. $P(S=s) = 0$ if $x+y \neq s$

Proof conti.

$$E[X+Y] = \sum_x \sum_y (x+y) P(x,y)$$

$$E[aX+bY+cZ] = ?$$

$$aE[X]+bE[Y]+cE[Z]$$

$$= \sum_x \sum_y x P(x,y) + \sum_x \sum_y y P(x,y)$$

$$= \sum_x x \sum_y P(x,y) + \sum_y \sum_x y P(x,y)$$

$$= \sum_x x \cdot P(x) + \sum_y y \sum_x P(x,y)$$

$$= E[X] + \sum_y y P(y)$$

$$= E[X] + E[Y]$$

Q. What's the value?

- What is $E[E[X] + 1]$?

$\boxed{\text{A.}}$ E[X]+1 B. 1 C. 0

$$E[\underbrace{E[X]+1}^{\text{const}}]$$

$$E[E[X]+1]$$

$$= E[E[X]] + E[1]$$

$$= E[X] + 1$$

Expected value of a function of X

- If $\boldsymbol{f}$ is a function of a random variable X, then $Y = \boldsymbol{f}(X)$ is a random variable too
- The expected value of $Y = \boldsymbol{f}(X)$ is

$$E[f(X)] = ?$$

$$E[Y] = \sum_y y\,P(y)$$

Expected value of a function of X

- If $\boldsymbol{f}$ is a function of a random variable X, then $Y = \boldsymbol{f}(X)$ is a random variable too

- The expected value of $Y = \boldsymbol{f}(X)$ is

$$E[Y] = E[f(X)] = \sum_x f(x)P(x)$$

The exchang of variable theorem

$$E[f(x)] = E[Y] = \sum y\, p(y)$$

if $\{x\} \to \{y\}$ is bijection $\quad p(y) = p(x)$

$$= \sum y\, p(x)$$

$$= \sum_{x} f(x)\, p(x)$$

if several x in $\{x_s\} \to$ one y_s value

$$E[Y] = \sum_{y \in 1} y\, p(x) + y_s\, p(y_s)$$

$$= \sum_{y \in 1} y\, p(x) + y_s \sum_{x \in \{x_s\}} p(x)$$

$$= \sum y\, p(x) = \sum_{x} f(x)\, p(x)$$

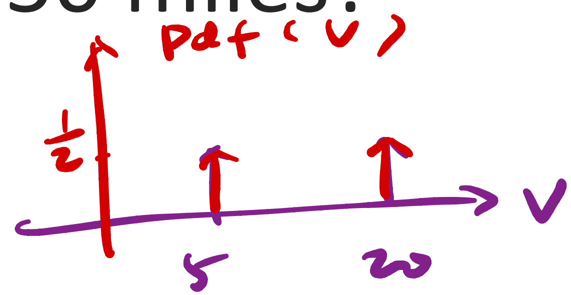
Expected time of cat

- A cat moves with random constant speed **V**, either 5mile/hr or 20mile/hr with equal probability, what's the expected time for it to travel 50 miles?

$f(E[v]) \neq E[f(v)]$

$E[T] = ?$

$\neq \frac{D}{E[V]}$



$T = \frac{50}{v} = f(v)$

$E[\frac{50}{v}] = \sum f(v) p(v)$

$v_1 = 5$

$v_2 = 20$

$= \frac{50}{v_1} \cdot p(v_1) + \frac{50}{v_2} p(v_2) = 6 \ldots$

after class

Q: Is this statement true?

If there exists a constant such that $\boldsymbol{P}(X \geq a) = 1$, then $E[X] \geq a$. It is:

A. True

B. False

Variance and standard deviation

- The variance of a random variable X is

$$\boxed{var[X] = E[(\underbrace{X - E[X])^2}_{f(x)}]}\; = E[f(x)]$$

- The standard deviation of a random variable X is

$$std[X] = \sqrt{var[X]}$$

Properties of variance

- For random variable X and constant k

$$var[X] \geq 0$$

$$var[kX] = k^2 var[X]$$

A neater expression for variance

- Variance of Random Variable X is defined as:

$$var[X] = E[(X - E[X])^2]$$

- It's the same as:

$$\boxed{var[X] = E[X^2] - E[X]^2}$$

A neater expression for variance

$$var[X] = E[(X - E[X])^2]$$

A neater expression for variance

$$var[X] = E[(X - E[X])^2]$$

$$var[X] = E[(X - \mu)^2] \quad where\ \mu = E[X]$$

A neater expression for variance

$$var[X] = E[(X - E[X])^2]$$

$$var[X] = E[(X - \mu)^2] \quad where\ \mu = E[X]$$

$$= E[X^2 - 2X\mu + \mu^2]$$

$$= E[X^2] - E[2X\mu] + E[\mu^2]$$

$$= E[X^2] - 2E[X\mu] + E[\mu^2]$$

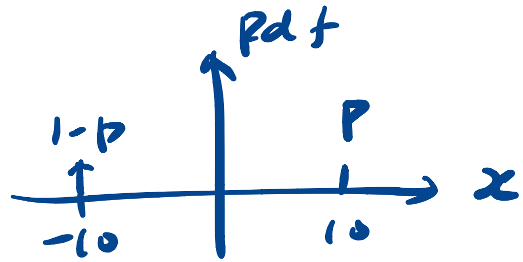
$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - E[X]^2$$

Variance: the profit example

- For the profit example, what is the variance of the return? We know E[X]= 20p-10

$$E[f(X)] = \sum_x f(x)P(x)$$



$$var[X] = E[X^2] - (E[X])^2$$

$$= \sum_x x^2 p(x) - (20p-10)^2$$

$$= 10^2 \cdot p + (-10)^2 (1-p) - (20p-10)^2$$

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

See you next time

See You!