Understanding Computation

Mathematics & Computation

Machines have helped with calculations for a long time

Can we use machines to reason too?

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Calculemus!

Formal Logic: Reasoning made into a calculation

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Formal systems based on axioms and logic:
for machines & modern mathematicians

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<u>Foundational problem:</u> How to choose one's axioms?

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They should not give rise to contradictions!

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Early 1900s: Crisis in mathematical foundations

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Formal systems based on axioms and logic: for machines & modern mathematicians

Foundational problem: How to choose one's axioms?

They should not give rise to contradictions!

Early 1900s: Crisis in mathematical foundations

Contradictions discovered while attempting to formalize notions involving infinite sets

David Hilbert

- 1928, Hilbert's Program:

  "Mechanize" mathematics

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- Finite set of axioms and inference rules. An algorithm to determine the truth of any statement

  Need to find a consistent & complete set of axioms

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  Need to find a consistent & complete set of axioms

- The system should also afford a proof of its own consistency
  - Based on "safe" axioms — i.e., axioms involving only finite objects — preferably

Mathematics & Computation

Mechanized math

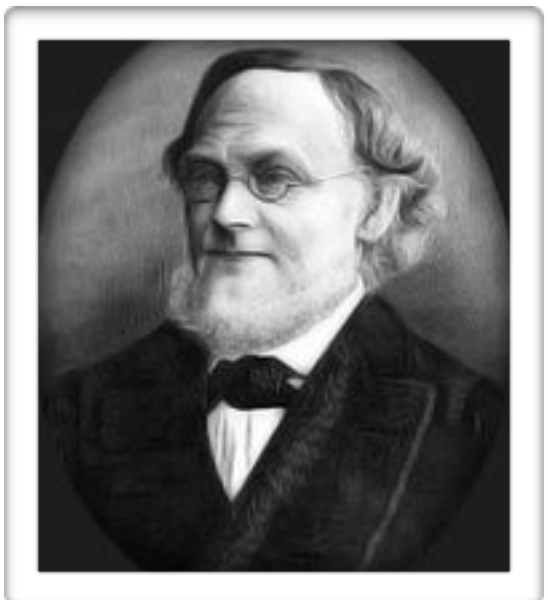
Beyond just philosophical interest!

Can resolve stubborn open problems

Replace mathematicians with mathe-machines!

Goldbach's Conjecture

Every even number > 2 is the sum of two primes



Letter from Goldbach to Euler dated 7 June 1742

Collatz Conjecture

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Program Collatz (n:integer)
  while n > 1 {
    if Even(n) then n := n/2
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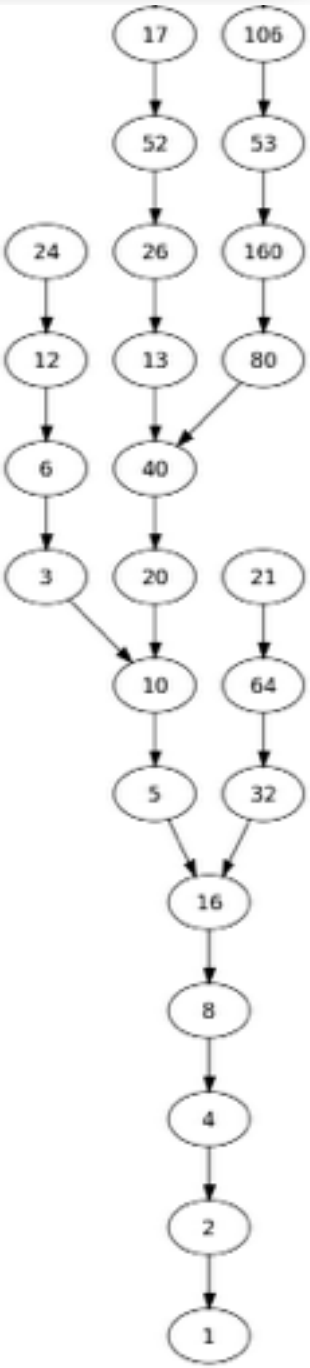

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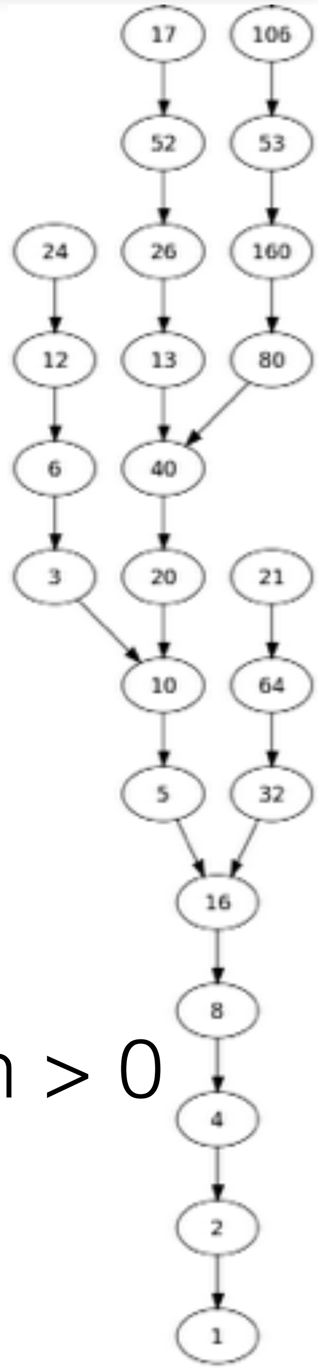

Conjecture: Collatz(n) halts for every n > 0

Kurt Gödel

- German logician, at age 25 (1931) proved:

"No matter what (consistent) set of axioms are used, a rich system will have true statements that can't be proved"

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- Shook the foundations of
  - mathematics
  - philosophy
  - science
  - everything

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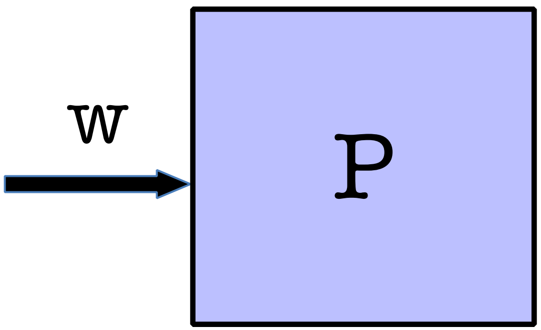
- British mathematician
  - cryptanalysis during WWII
  - arguably, father of AI, CS Theory
  - several books, movies
- Mathematically defined computation
  - and proved (1936) that **The Halting Problem** has no general algorithm

Halting Problem

- Given program `P`, input `w`:



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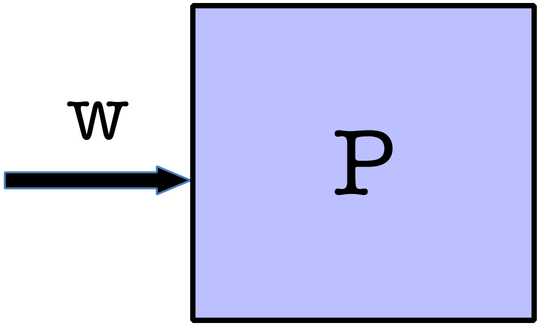

Will `P(w)` halt?

Why would we care about the Halting Problem?

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Program P()
    n := 4
    forever:
        if found-two-primes-that-sum-to(n)
        then n := n + 2
        else halt
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```

Does P halt ? ⟵ **Solves Goldbach conjecture!**

Why would we care about the Halting Problem?

Does <u>Find-proof</u> halt on w? ≡ Is w a provable theorem?

w

```
Program Find-proof(w)

p := empty-string
forever
    p := successor(p)
    if Verify-proof(w,p)
    then halt
```

Alas!

Alas!

There is no program that solves the Halting Problem!

No use trying to find one!

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How can there be problems that can't be solved?

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How can there be problems that can't be solved?

What is a problem? What is a program?

Computation

Problem:
To compute a function F that maps each input (a string) to an output bit

Program:
A finitely described process taking a string as input, and outputting a bit (or not halting)

`P` solves F if for every x, `P(x)` outputs F(x) and halts

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Enough to model *interactive* computation too:
`P*(x,state)` outputs `(y,new_state)`

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- Programs can be *enumerated* — listed sequentially — (say, lexicographically) so that every program appears somewhere in the list

1	ε
2	0
3	1
4	00
5	01
6	10
7	11
8	000
9	001
10	010
11	011
12	100

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- A program is a finite bit string
- Programs can be *enumerated* — listed sequentially — (say, lexicographically) so that every program appears somewhere in the list
- The set of all programs is countable.

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- A function assigns a bit to each finite string
- Corresponds to an infinite bit string
- The set of all functions is uncountable!
  - As numerous as, say, real numbers in [0,1]

1	ε	***0***
2	0	***0***
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There are uncountably many functions!

But only countably many programs

Almost every function is uncomputable!

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Proving that there are uncountably many real numbers:
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Proving that there are uncountably many real numbers: "**Diagonalization**" argument by Cantor

Showing Halting Problem to be uncomputable: a similar argument (*later*)

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If the Halting Problem can be reduced to F
then F must be uncomputable!

Post Correspondence Problem

<u>Theorem</u> [Post'46]: Halting Problem (formulated for "Turing Machines") reduces to <u>PostCP</u> — a "combinatorial" problem

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Given: Dominoes, each with a top-word and a bottom-word

b	**ba**	**abb**	**abb**	**a**
bbb	**bbb**	**a**	**baa**	**ab**

Can one arrange them (using <u>any number of copies of each type</u>) so that the top and bottom strings are identical?

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Typically, easier than reducing Halting Problem directly to F

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Many more *interesting* problems:
http://en.wikipedia.org/wiki/List_of_undecidable_problems

Induction

Inductive Proofs

- Example: How many “moves” to assemble a jigsaw puzzle?
  - move = join two *clumps*
  - *clump* = connected pieces
  - only successful moves count

- **Theorem**: It takes exactly n-1 moves to assemble an n-piece jigsaw puzzle (irrespective of which moves)

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Assume any (n-1)-piece puzzle requires n-2 moves

Consider any n-piece puzzle:

n-2 moves for all but last

One more move for last

total = (n-2)+1 = n-1

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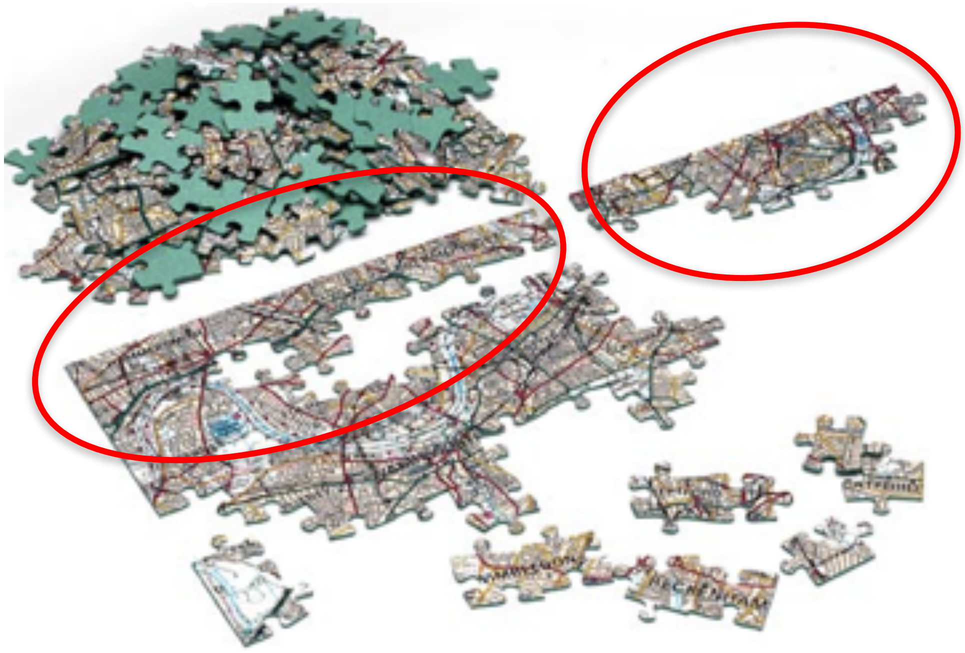

Last move could join two large clumps

Inductive Proofs

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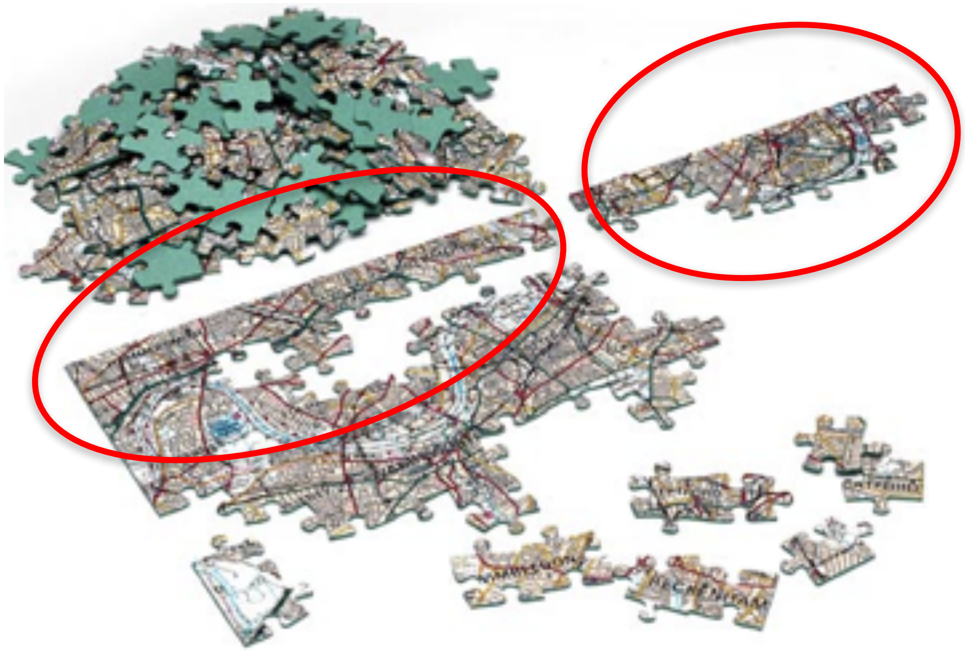

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The argument presented implicitly assumes puzzle is built piece-by-piece

Induction Template

- Base Case: Let n = ⟨**some small values**⟩.
  Then ⟨**show claim holds for n**⟩

- Induction Step: Consider any *arbitrary* integer n ⟨**greater than base-case values**⟩.

  Induction hypothesis: Assume that for all integers k < n (and k ≥ ⟨**smallest value**⟩), ⟨**claim holds for k**⟩

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May need a stronger claim than originally asked to prove

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The clever stuff. Be careful to consider *arbitrary instance* of size n. Relate it to one or more instances for which IH is assumed.

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Stronger Claim: Any ***clump*** with n pieces takes exactly n-1 moves to assemble

Example

- Base Case: Let n = **1**.
  Then, **any <u>clump</u> with n pieces is just a single piece, and it needs 0 = n-1 moves to assemble**

- Induction Step: Consider any *arbitrary* integer n **> 1**.

  Induction hypothesis: Assume that for all integers k < n (and k ≥ **1**), **any clump with k pieces needs k-1 moves to assemble**

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  Consider an *arbitrary* clump with n pieces, and an *arbitrary* sequence of moves to assemble it.
  - Last move joins 2 clumps of size k and n-k, where 1 ≤ k < n.
  - By IH, the two clumps took k-1 and n-k-1 moves each.
  - Overall (k-1) + (n-k-1) + 1 = n-1 moves. ✅

Simple non-inductive proof

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    - m moves reduce it by m
  - Initially, n clumps (each of one piece)
  - At the end, 1 clump (of all pieces)
  - Therefore, if m moves overall, 1 = n - m.
  - Hence m = n - 1

If you came in late:

- https://courses.engr.illinois.edu/cs374/
- Immediately join Piazza
- Immediately check access to Moodle

Links to Piazza and Moodle are on course home page