Limitations of Finite Automata

Lecture 6

Today

Proving that certain languages need DFAs with a large number of states

Proving that certain (“easy”) languages cannot be decided using DFAs at all!

Using Closure Properties of regular languages to reason about non-regularity

Need for Memory

e.g., Language $L = \{\ 0^{33}\ \}$ (just one string, of 33 0's)

"Clearly" a DFA for L will need to keep count of the 0's seen, up to 33

35 states (count = 0,1,…,33 and "crashed")

How do we rule out the possibility of a clever DFA with fewer states?

Need for Memory

Suppose M with $d < 35$ states s.t. $L(M) = L = \{ 0^{33} \}$

Consider $\delta^*(s,w)$ for $w \in \{ 0^i \mid i \in [0,34] \}$

Pigeonhole Principle $\Rightarrow$ at least two values $i < j$ s.t.

$\delta^*(s,0^i) = \delta^*(s,0^j) = q$ (say)

Let $u = 0^i$ and $v = 0^j$

Let $x = 0^k$ where $k = 33 - i$ ($k \geq 0$ since $i < j \leq 34$).

$ux \in L \Rightarrow \delta^*(s, ux) \in F$ and $vx \notin L \Rightarrow \delta^*(s, vx) \notin F$

But $\delta^*(s, ux) = \delta^*(q, x) = \delta^*(s, vx)$!

Abstracting the proof

Consider $\delta^*(s,w)$ for
$w \in \{ 0^i \mid i \in [0,34] \}$

By the pigeonhole principle,
at least two values $i < j$ s.t.
$\delta^*(s,0^i) = \delta^*(s,0^j) = q$ (say)

Let $u = 0^i$ and $v = 0^j$

Let $x = 0^k$ where $k = 33 - i$
($k \geq 0$ since $i < j \leq 34$).

$ux \in L \Rightarrow \delta^*(s, ux) \in F$ and
$vx \notin L \Rightarrow \delta^*(s, vx) \notin F$

But $\delta^*(s, ux) = \delta^*(q, x) = \delta^*(s, vx)$
!

Come up with a set S, $|S| = d$

s.t. for *any two distinct* $u, v \in S$

$\exists\, x\ s.t$

$|\{ ux , vx \} \cap L| = 1$

Proves that any DFA for L must have at least d states

Abstracting the proof as a Game!

SuvxGame(L,d)

Defender (claim: ∃ DFA for L with $< d$ states)		Challenger (claim: no DFA for L has $< d$ states)
	S ←	Picks a set $S \subseteq \Sigma^*$, $\|S\| \geq d$
Picks distinct $u,v \in S$	u,v →	
	x ←	Picks $x \in \Sigma^*$

Challenger wins if $|\{ux,vx\} \cap L| = 1$. Else Defender wins

Theorem : If there is a DFA for L with $< d$ states, then Defender has a winning strategy in SuvxGame(L,d)

$\forall S \subseteq \Sigma^*, |S| \geq d, \ \exists u,v \in S, u \neq v, \ \ \forall x \in \Sigma^*, \ \ ux \in L \Leftrightarrow vx \in L$

Strings Indistinguishable to a DFA

Let $M = (\Sigma, Q, \delta, s, F)$ be an arbitrary DFA

Suppose $\delta^*(s,u) = \delta^*(s,v) = q$

Then, for every x, $\delta^*(s,ux) = \delta^*(s,vx) = \delta^*(q,x)$

i.e., M "can't tell the difference" between having seen u and v

In particular, for every x, $ux \in L(M) \Leftrightarrow vx \in L(M)$ $\qquad (*)$

Theorem : If there is a DFA for L with $< d$ states, then Defender has a winning strategy in SuvxGame(L,d)

Proof : Winning strategy: Let $L=L(M)$, $M = (\Sigma, Q, \delta, s, F)$, $|Q| < d$. Receive S. Since $|S| > |Q|$, by pigeonhole principle, $\exists\, u,v \in S$, s.t., $\delta^*(s,u)=\delta^*(s,v)$. Send (u,v). By $(*)$, Challenger can't win the game.

Proving Lowerbound on DFA size

Theorem : If there is a DFA for L with $< d$ states, then
Defender has a winning strategy in SuvxGame(L,d)

$\forall S \subseteq \Sigma^*, |S| \geq d, \; \exists u,v \in S, u \neq v, \; \forall x \in \Sigma^*, \; ux \in L \Leftrightarrow vx \in L$

$\exists S \subseteq \Sigma^*, |S| \geq d, \; \forall u,v \in S, u \neq v, \; \exists x \in \Sigma^*, \; |\{ux,vx\} \cap L| = 1$

Corollary : If the Challenger has a winning strategy in the SuvxGame(L,d) (*so that the Defender doesn't*), then there is no DFA for L with $< d$ states.

To prove that L has no DFA with $< d$ states, enough to show a winning strategy for the Challenger in SuvxGame(L,d)

Fooling set of size d

Fooling Set S: $\forall u,v \in S, u \neq v, \; \exists x \in \Sigma^*, \; |\{ux,vx\} \cap L| = 1$

Example

$L_k = \{ w \mid w \text{ ends in } 1(0+1)^{k-1} \}$

An NFA with k+1 states: $Q = \{0,1,\ldots,k\}$, $s = 0$, $F = \{k\}$.
$\delta(0,0) = \{0\}$, $\delta(0,1) = \{0,1\}$. $\delta(i,b) = \{i+1\}$ for $1 \leq i < k$ and $b \in \{0,1\}$.

DFA? Intuitively, need to remember last k bits, as any of them could turn out to be at the k^{th} position from end. DFA with a state for each possible prefix: $(1+1+2+4+\ldots+2^{k-1}) = 2^k$ states.

Claim: Any DFA for L_k must have at least $d = 2^k$ states!

Winning strategy for challenger?
$S = \{0,1\}^k$. For any u,v, let x be as follows.

There is some position where u, v disagree. Say i^{th} position from right ($1 \leq i \leq k$). Let $x = 0^{k-i}$. Then ux, vx disagree at the k^{th} position from right. Hence, exactly one of them will be in L_k.

Non-Regular Languages

e.g., Language $L = \{ 0^n 1^n \mid n \geq 0 \}$

"Clearly" an automaton will need to count the number of 0's and match it against the number of 1's (if it is scanning the input bit by bit)

Cannot do that in a DFA

How do we prove it?

Show an infinite fooling set!

The Game

dSuvxGame(L)

Defender (claim: ∃ DFA for L)		Challenger (claim: no DFA for L)
Picks	d →	
	← S	Picks a set $S \subseteq \Sigma^*$, $\lvert S\rvert \geq d$
Picks distinct $u,v \in S$	u,v →	
	← x	Picks $x \in \Sigma^*$

Challenger wins if $\lvert\{ux,vx\} \cap L\rvert = 1$. Else Defender wins

Theorem : If there is a DFA for L, then Defender has a winning strategy in dSuvxGame(L)

$\exists d \in \mathbf{Z}^+\ \forall S \subseteq \Sigma^*, |S| \geq d,\ \exists u,v \in S, u \neq v,\ \forall x \in \Sigma^*,\ ux \in L \Leftrightarrow vx \in L$

Proving no DFA

Theorem : If there is a DFA for L, then the Defender has a *winning strategy* in dSuvxGame(L)

$\exists d \in \mathbf{Z}^+ \; \forall S \subseteq \Sigma^*, |S| \geq d, \; \exists u,v \in S, u \neq v, \;\; \forall x \in \Sigma^*, \;\; ux \in L \Leftrightarrow vx \in L$

$\forall d \in \mathbf{Z}^+ \; \exists S \subseteq \Sigma^*, |S| \geq d, \; \forall u,v \in S, u \neq v, \;\; \exists x \in \Sigma^*, \; |\{ux,vx\} \cap L| = 1$

Corollary : If the Challenger has a winning strategy in the dSuvxGame(L) (so that the Defender doesn't), then there is no DFA for L.

To prove that L has no DFA, enough to show a winning strategy for the Challenger in dSuvxGame(L)

an infinite Fooling set

Fooling Set S: $\forall u,v \in S, u \neq v, \;\; \exists x \in \Sigma^*, \; |\{ux,vx\} \cap L| = 1$

Non-Regularity: Examples

- $L = \{ 0^n 1^n \mid n \geq 0 \}$
  - Winning strategy for challenger?
  - $S = \{0\}^*$. For any u,v, let $x = 1^{|u|}$ so that $ux \in L$, $vx \notin L$.
- $L = \{ w \mid w$ has equal number of 0s and 1s $\}$
  - Same winning strategy as above
- $L = \{ 0^p \mid p$ is a prime number $\}$
  - Same fooling set as above! $S = \{0\}^*$
  - Rest of the strategy?

Non-Regularity: Examples

- $L = \{\ 0^p \mid p$ is a prime number $\}$

Fun exercise!
Hint: *primorial*

- Fooling set $S = \{0\}^*$. Rest of the strategy?
  - Given $u{=}0^i$, $v{=}0^j$, (say, $i < j$, w.l.o.g.) find a non-negative number k s.t. exactly one of $i{+}k$ and $j{+}k$ is prime. Then, let $x{=}0^k$.
  - Solution 1: **Fact**: We can find arbitrarily large gaps between successive prime numbers, for arbitrarily large prime numbers.

    Let $\Delta = j - i$. Let p_1 and p_2 be successive primes s.t. $p_1 \geq i$ and $p_2 - p_1 > \Delta$. Let $k = p_1 - i$. Then $k{+}i = p_1$ is prime, $k{+}j = p_1 + \Delta < p_2$ is not.
  - Solution 2: Let $\Delta = j - i$. Let $p \geq i$ be a prime, Note that $p{+}r\Delta$ is prime for $r{=}0$, and non-prime for $r = p$. Hence $\exists$ some $r \in [0, p]$ s.t. $p{+}r\Delta$ is prime but $p + (r{+}1)\Delta$ is a non-prime. Set $k = p{+}r\Delta - i$.

Non-Regularity via Closure

Recall: Several operations on languages that preserve regularity

If L_1, L_2 regular, so is L_1 **op** L_2

Gives a way to prove non-regularity!

Suppose we already knew that L_2 is regular, but L_1 **op** L_2 is not.

Then L_1 is not regular!

Non-Regularity: Examples

- Let $L = \{ w \mid w$ has unequal number of 0s and 1s $\}$.
  - We know that $\bar{L}$ is not regular. Hence L can't be.
- Let $L = \{ 0^n1^n \mid n \geq 0 \} \cup \{ w \mid |w|$ is odd $\}$
  - Let $L' = \{0^n1^n \mid n \geq 0 \}$ and $L'' = \{w \mid |w|$ is odd $\}$.
    Then $L' = L - L''$. Now, L'' is regular. If L regular, L' will be too!
- Let $L_1 = \{ 0^n1^n \mid n \geq 0 \}$. Let $L = \{ w \mid w \in L_1$ and $|w| \equiv 0 \pmod 3) \}$.
  Prove that L is not regular.
  - Use $L = L_1 \cap L_2$ where $L_2 = \{ \mathrm{w} \mid |w| \equiv 0 \pmod 3 \}$?
  - **No!** (L_1 not regular, L_2 regular) $\not\Rightarrow$ ($L_1 \cap L_2$ not regular) !
    - e.g., L_2 is a finite set

Non-Regularity via Closure

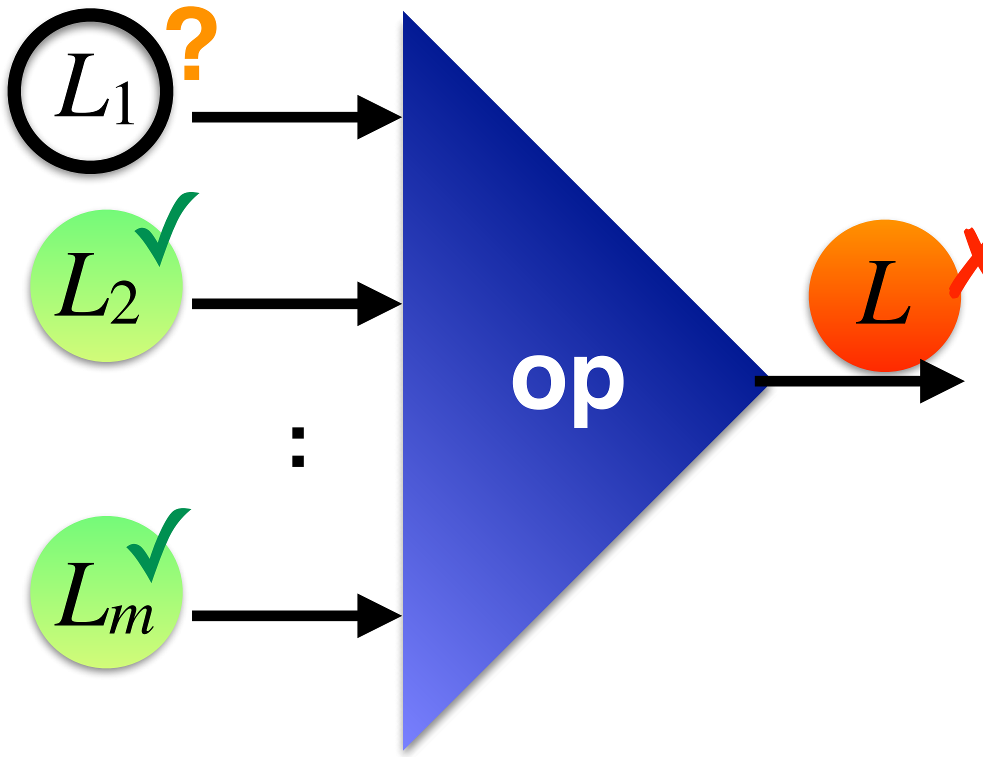
Recall: Several operations on languages that preserve regularity

If L_1, L_2 regular, so is L_1 **op** L_2

Gives a way to prove non-regularity!

Suppose we already knew that L_2 is regular, but L_1 **op** L_2 is not.

Then L_1 is not regular!



Non-Regularity: Examples

- Let $L_1 = \{ 0^n 1^n \mid n \geq 0 \}$. Let $L = \{ w \mid w \in L_1$ and $|w| \equiv 0 \pmod 3) \}$. Prove that L is not regular.
  - Let $L' = \{0\}L\{1\}$. and $L'' = \{00\}L\{11\}$.
  - If L is regular, so are L' and L''
  - But $L_1 = L \cup L' \cup L''$
  - Since L_1 not regular, L is not regular
- Or directly use the fooling set argument: e.g., $S = \{ 0^{3i} \mid i \geq 0\}$

How to Pick a Fooling Set

Make sure you don't put in any two strings which can let the defender win!

In particular, all the strings you include (except maybe one) should be prefixes of strings in the language

e.g., for $L = \{ 0^n 1^n \mid n \geq 0 \}$ don't include 1 and 10 (say)

Intuitively, each prefix in the fooling set has a different value for a parameter that a machine will need to remember

Non-Regular
⇒ Infinite Fooling Set?

We saw that infinite fooling set ⇒ non-regular. Converse?

(Will this proof strategy always work, in principle?
Or maybe for some non-regular languages the Defender has a winning strategy in dSuvxGame?)

Myhill-Nerode theorem

Optional Reading

Converse does hold!
(Defender has a winning strategy only if L is regular.)

But not necessarily easy (or even possible) to compute the winning strategy from say, an English description.

FYI: Another Game

PumpingGame(L)

Defender (claim: ∃ DFA for L)		Challenger (claim: no DFA for L)
Picks	d →	
	← w	Picks $w \in L$, $\lvert w\rvert \geq d$
$x,y,z \in \Sigma^*$ s.t. $w=xyz$, $\lvert xy\rvert \leq d$, $\lvert y\rvert \geq 1$	x,y,z →	
	← i	Picks $i \in \mathbf{N}$
Challenger wins if		

Theorem : If there is a DFA for L, then <u>Defender has a winning strategy</u> in PumpingGame(L). Hence if Challenger has a winning strategy, L not regular. [Converse doesn't hold.]