Finite State Machines

Lecture 3

Recall a Language is Regular if

- L is empty
- L contains a single string (could be the empty string)
- If L_1, L_2 are regular, then $L = L_1 \cup L_2$ is regular
- If L_1, L_2 are regular, then $L = L_1 L_2$ is regular
- If L is regular, then $L*$ is regular

Unbounded vs. Infinite

- Why do we need bullet 5?
- Why can't we say that L^* is the infinite union of $\{\varepsilon\} \cup L \cup LL \cup LLL \cup \ldots$
- Recursive definitions: at every branch of recursion we need to reach a base case in **finite number** of steps.
- We can invoke the union rule for any integer n number of steps
- infinity is not a number! I can only produce infinite sets by an operation like the *.

Complexity of Languages

Central Question: How complex an algorithm is needed to compute (aka decide) a language? How much memory do I need?

Today: a simple class of algorithms, that are fast and can be implemented using minimal hardware

Finite State Machines -Deterministic Finite Automata (FSM-DFA)

DFAs around us: Vending machines, Elevators, Digital watch logic, Calculators, Lexical analyzers (part of program compilation), …

Multiple of 5

```
MULTIPLEOF5(w[1..n]):
    rem ← 0
    for i ← 1 to n
        rem ← (2 · rem + w[i]) mod 5
    if rem = 0
        return TRUE
    else
        return FALSE
```

- Could do long division, keep the intermediate results in an array but I don't want to spend that much memory!
- Only one variable, rem, which represents the remainder of the part of the string I read so far when I divided by 5.

Multiple of 5

m0=2m if I see “0” next
m1=2m+1 if I see “1” next

<u>MULTIPLEOF5($w[1..n]$):</u>

```
rem ← 0
for i ← 1 to n
    rem ← (2 · rem + w[i]) mod 5
if rem = 0
    return TRUE
else
    return FALSE
```

- If I know the remainder for m mod 5, and I read one more bit then line 3 tells me what the new remainder is (either m0 or m1)

Multiple of 5

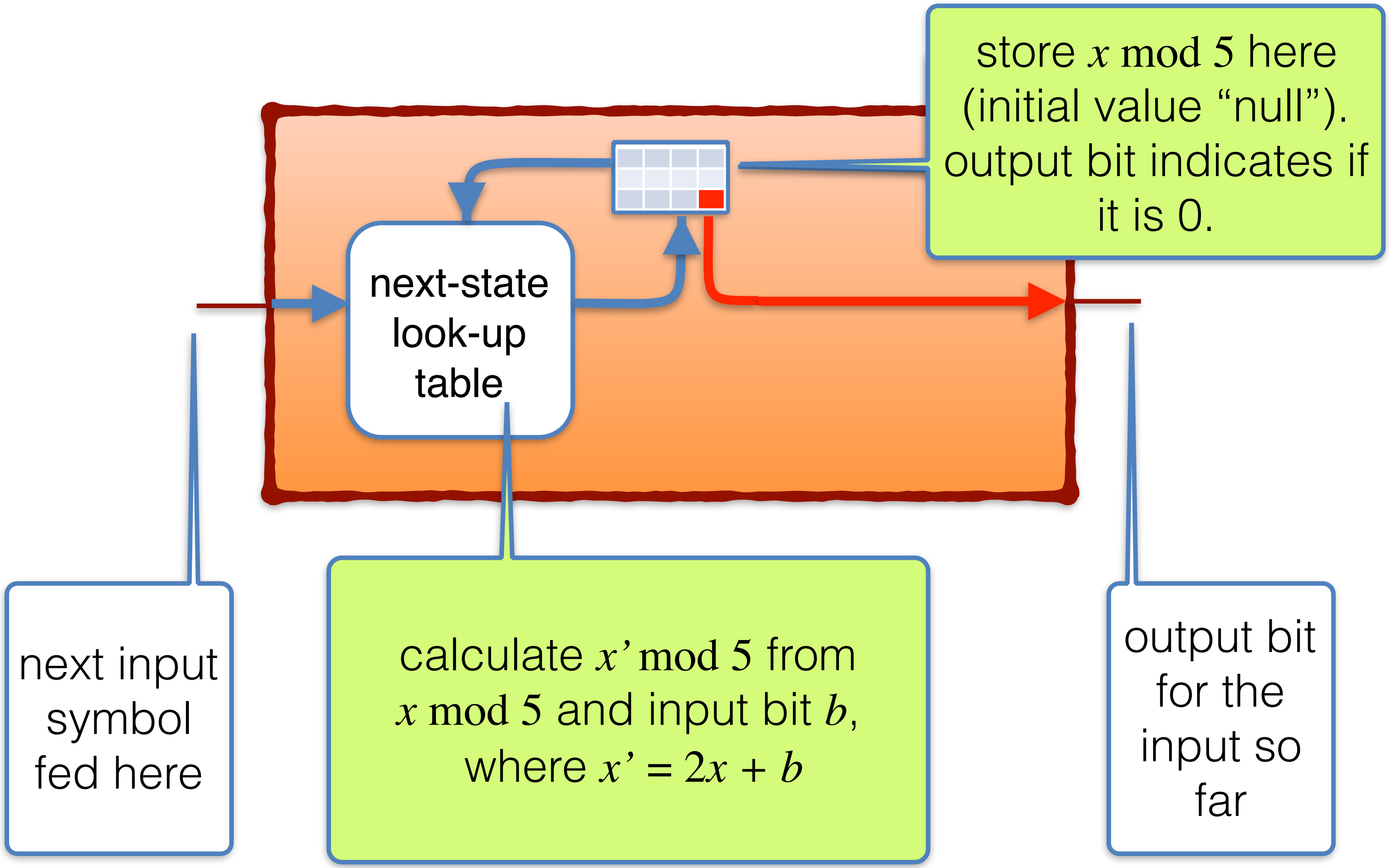
- Important feature of algorithm: Aside from variable i which counts the input bits and is necessary to read input, I only have one variable rem, which takes only a small (5) number of values.

- Streaming algorithm : Data flies by! Once w[i] is gone, it is gone forever.

- Variable has a very small number of states, which I am able to specify at compile time. Very small amount of memory!

DFA (a.k.a. FSM)

check if binary input is a multiple of 5



“Lookup” table

$$
\begin{array}{l}
\underline{\textsc{DoSomethingCool}(w[1\,..\,n]):} \\
\quad q \leftarrow 0 \\
\quad \text{for } i \leftarrow 1 \text{ to } n \\
\qquad q \leftarrow \delta[q, w[i]] \\
\quad \text{return } A[q]
\end{array}
$$

- q encapsulates the state of the algorithm
- Takes a small amount of values, which I know up front (e.g. q is a number between 1 and 4). Unbounded, not infinite!
- Depending on the character I read at position i, I change my state with function called delta (δ).
- I have a hardcoded array A and based on what the state is when I finish reading the string, I output the value of the array.

"Lookup" table

```
DoSomethingCool(w[1..n]):
    q ← 0
    for i ← 1 to n
        q ← δ[q, w[i]]
    return A[q]
```

If we want to use our new DoSomethingCool algorithm to implement MultipleOf5, we simply give the arrays δ and A the following hard-coded values:

q	$\delta[q, 0]$	$\delta[q, 1]$	$A[q]$
0	0	1	True
1	2	3	False
2	4	0	False
3	1	2	False
4	3	4	False

only one accepting state!

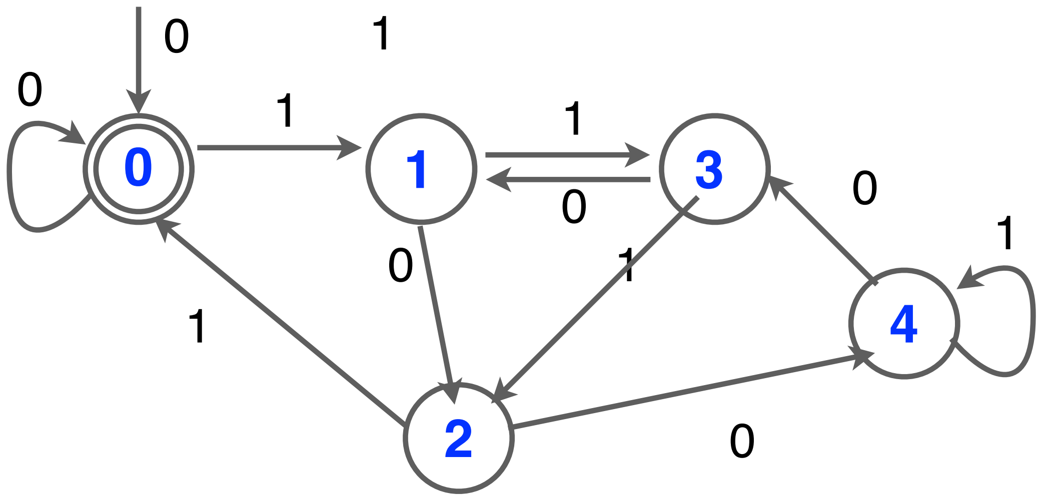
Instead of doing arithmetic at all, I could just **hard code** this lookup table into the code and simply do a lookup

DFA (a.k.a. FSM)

- Algorithm or Machine? Algorithm is a Machine!!
- Once you program the machine, you don't have to monitor it. It runs AUTOMATICALLY (Automaton…)

DFA (a.k.a. FSM)

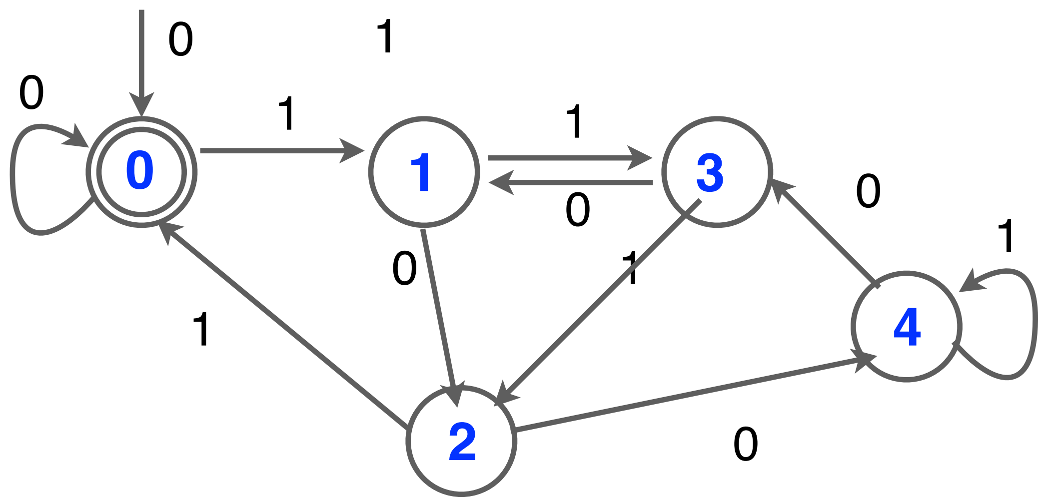
- Equivalent view as a graph!



DFA (a.k.a. FSM)

Example: check if input 01010101 is a multiple of 5

input bit	current state	next state
0	0	0
1	0	1
0	0	2
1	2	0
0	0	0
1	0	1
0	1	2
1	2	0



DFA (a.k.a. FSM)

check if input (MSB first) is a multiple of 5

input bit	current state	next state
0	0	0
1	0	1
0	0	2
1	2	0
0	0	0
1	0	1
0	1	2
1	2	0

How to fully specify a DFA (syntax):

FINITE Alphabet: Σ

FINITE Set of States: Q

Start state: $s \in Q$

Set of Accepting states: $A \subseteq Q$

Transition Function: $\delta : Q \times \Sigma \to Q$

$$\boxed{\delta(q,a) = (2q+a) mod 5}$$

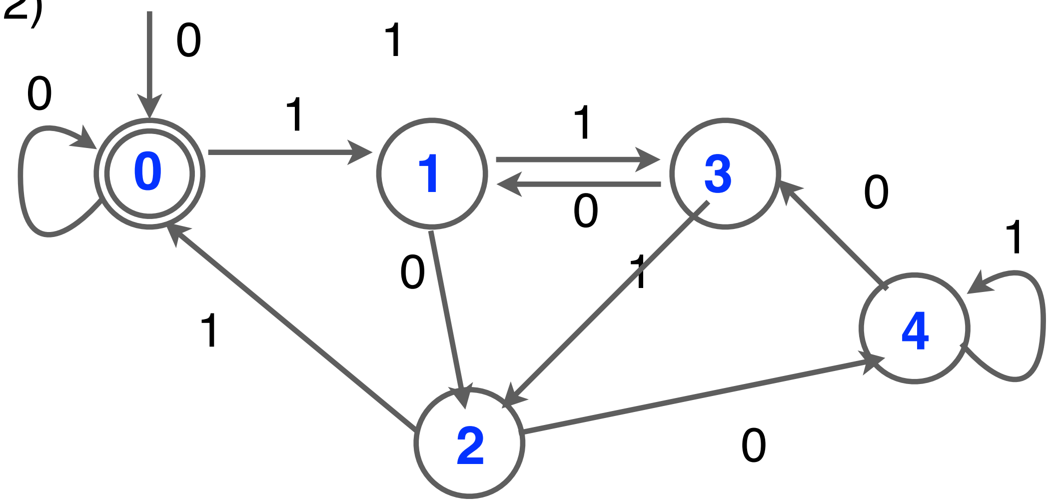
DFA (a.k.a. FSM)

3 equivalent ways to specify a FSM:

1)

q	$\delta[q, 0]$	$\delta[q, 1]$	$A[q]$
0	0	1	TRUE
1	2	3	FALSE
2	4	0	FALSE
3	1	2	FALSE
4	3	4	FALSE

2)



3)

$$\delta(q, a) = (2q + a) mod 5$$

Together with a description of what are the states and what are the accepting states

How to interpret these functions?

$$M = (\Sigma, Q, \delta, s, F)$$

- $\delta^*(q,w)$ be the state M reaches starting from a state $q \in Q$, on input $w \in \Sigma^*$
- *Recursive definition?*
- *What are the cases going to be?*

Behavior of a DFA on an input

$$M = (\Sigma, Q, \delta, s, F)$$

- $\delta^*(q,w)$ be the state M reaches starting from a state $q \in Q$, on input $w \in \Sigma^*$

- Formally,

  - $\delta^*(q,w) = q \;\; \text{if } w = \varepsilon$
  - $\delta^*(q,w) = \delta^*(\delta(q,a), x) \;\; \text{if } w = ax$

recursion!

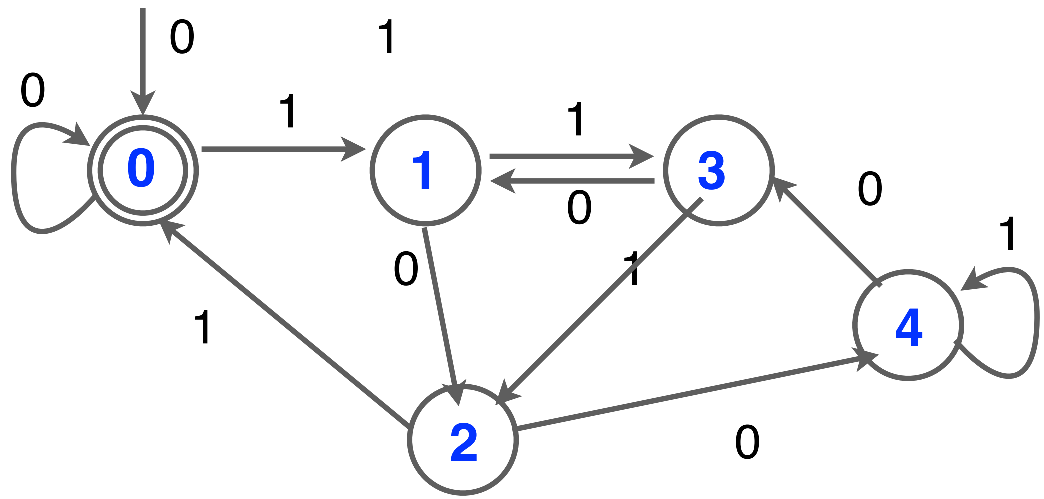
Behavior of a DFA on an input

$\delta^*(\mathbf{0}, 01001) = \mathbf{4}$

$\delta^*(\mathbf{0}, \varepsilon) = \mathbf{0}$

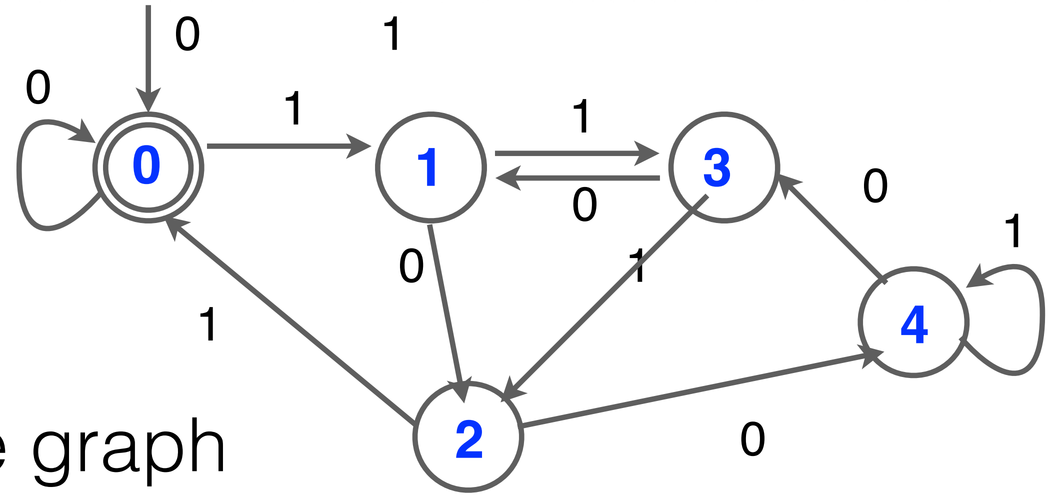
$\delta^*(\mathbf{0}, 010) = \mathbf{2}$

$\delta^*(\mathbf{2}, 01) = \mathbf{4}$



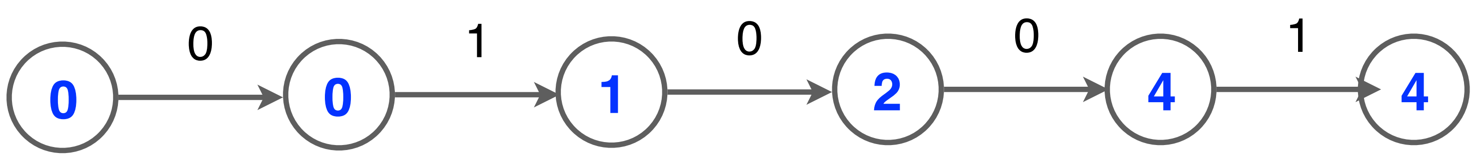
Behavior of a DFA on an input

$\delta^*(\mathbf{0}, 01001) = 4$



- Specify a walk in the graph

- Best represented as



Example: What strings does this machine accept?

Alphabet: $\Sigma = \{0,1\}$

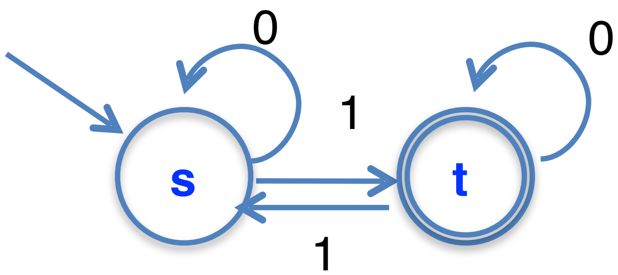
Set of States: $Q = \{s,t\}$

Start state: $s \in Q$

Accepting state: $t \in Q$

Transition Function: $\delta : Q \times \Sigma \rightarrow Q$

$\delta(s,0)=s,\ \delta(s,1)=t,\ \delta(t,0)=t,\ \delta(t,1)=s$



Question: what is L(M)?

Answer: strings with odd number of ones!

Input Accepted by a DFA

We say that M **accepts** $w \in \Sigma^*$ if M, on input w, starting from the start state s, reaches a final state

i.e., $\delta^*(s,w) \in F$

$L(M)$ is the set of all strings accepted by M

i.e., $L(M) = \{ w \mid \delta^*(s,w) \in F \}$

Called the language accepted by M

Input Accepted by a DFA

What kind of language is accepted by FSM?

- Automatic (it is an automaton after all)!

- We will use: REGULAR (not a coincidence)

Language is regular iff

-it is accepted by a finite state automaton

-it is described by a regular expression

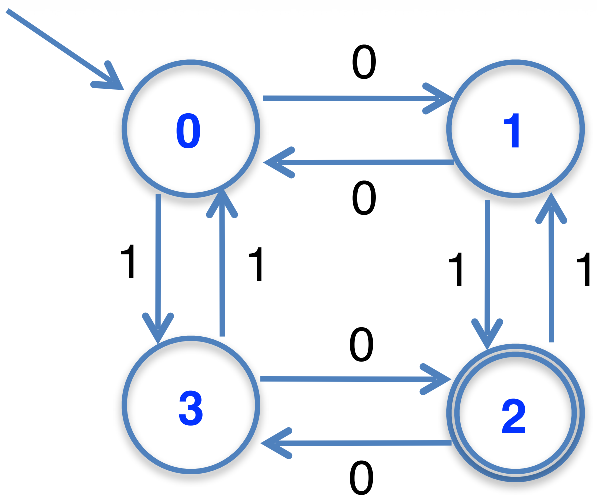
Warning

"M accepts language L" does not mean simply that M accepts each string in L.

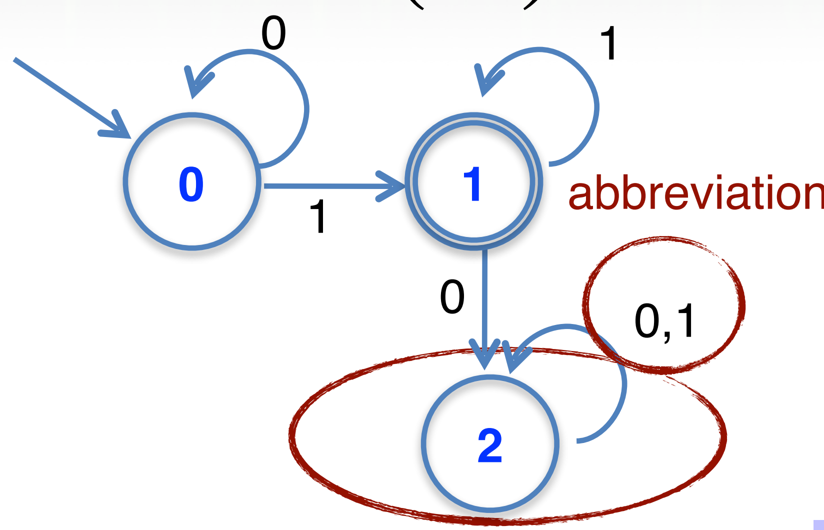
"M accepts language L" means M accepts each string in L and no others!

$$L(M) = L$$

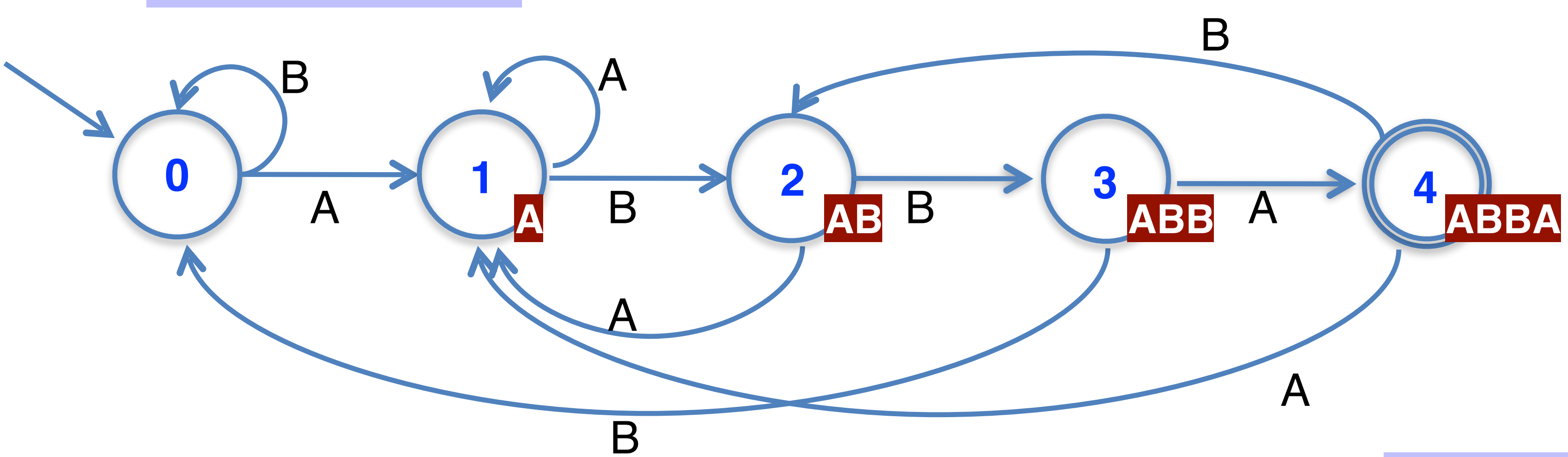
Examples: What is $L(M)$?



odd #0 and odd #1



0*11*



(A+B)*ABBA

Building DFAs

State = Memory

First, decide on Q

The state of a DFA is its entire memory of what has come before

The state must capture enough information to complete the computation on the suffix to come

When designing a DFA, think “what do I need to know at this moment?” That is your state.

Construction Exercise

$L(M) = \{w \mid w$ contains 00 $\}$

Is it regular??

$(0+1)^*00(0+1)^*$

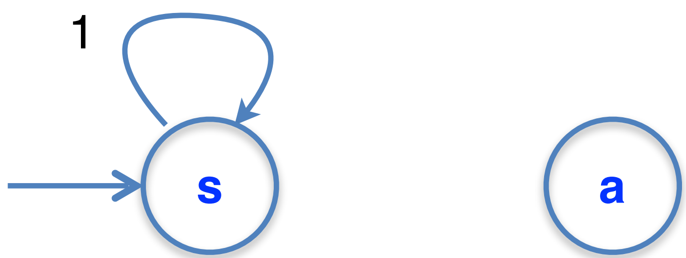
What should be in the memory?

Construction Exercise

$L(M) = \{w \mid w$ contains 00 $\}$

Is it regular?? (0+1)*00(0+1)*

What should be in the memory?



Construction Exercise

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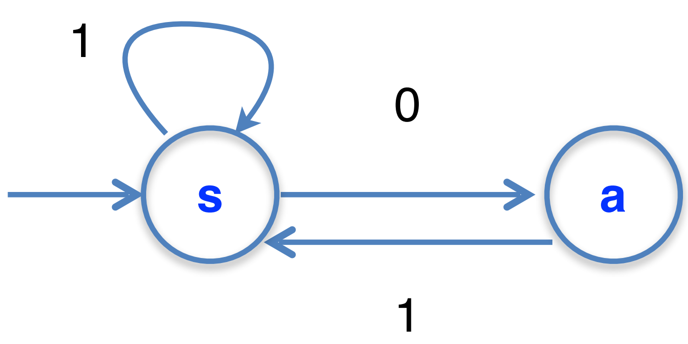

Construction Exercise

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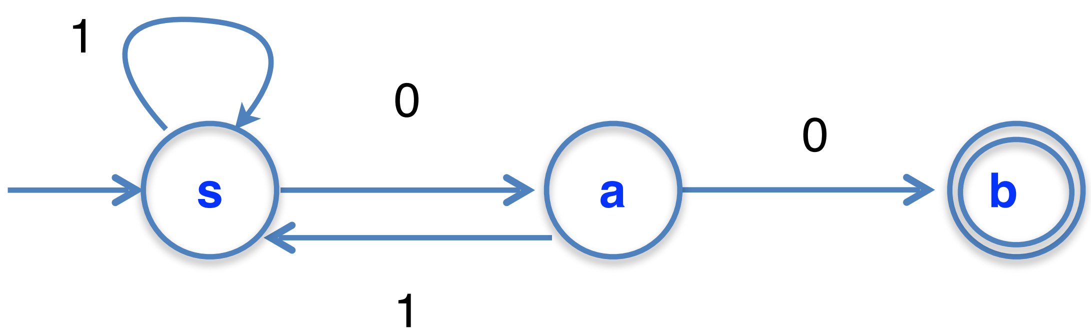

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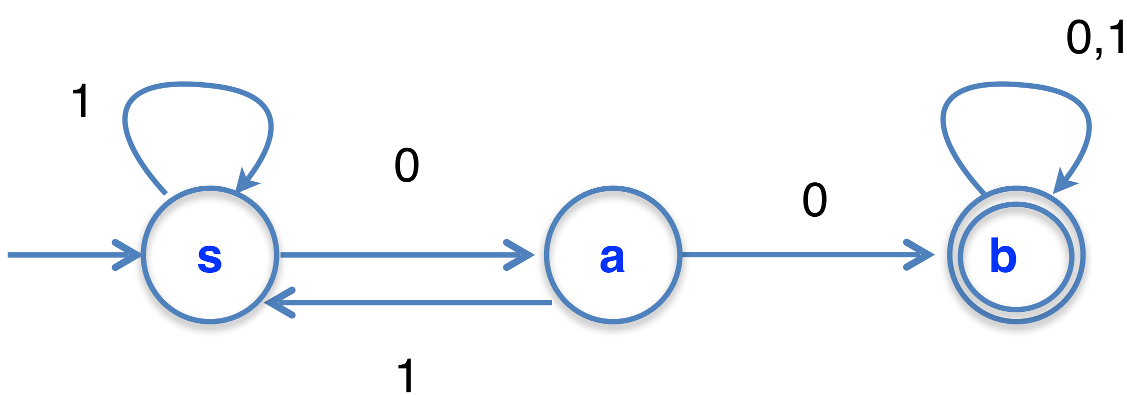


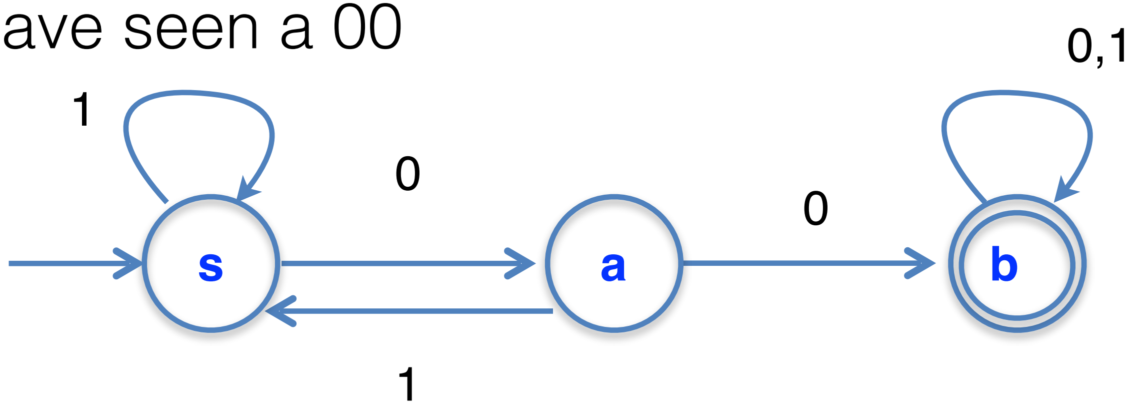
Construction Exercise

$L(M) = \{w \mid w \text{ contains } 00\}$

- s : I haven't seen a 00, previous symbol was not 0

- a: I haven't seen a 00, previous symbol was a 0

- b: I have seen a 00



- We have exhausted of all strings. Either accepted (with 00) or not.