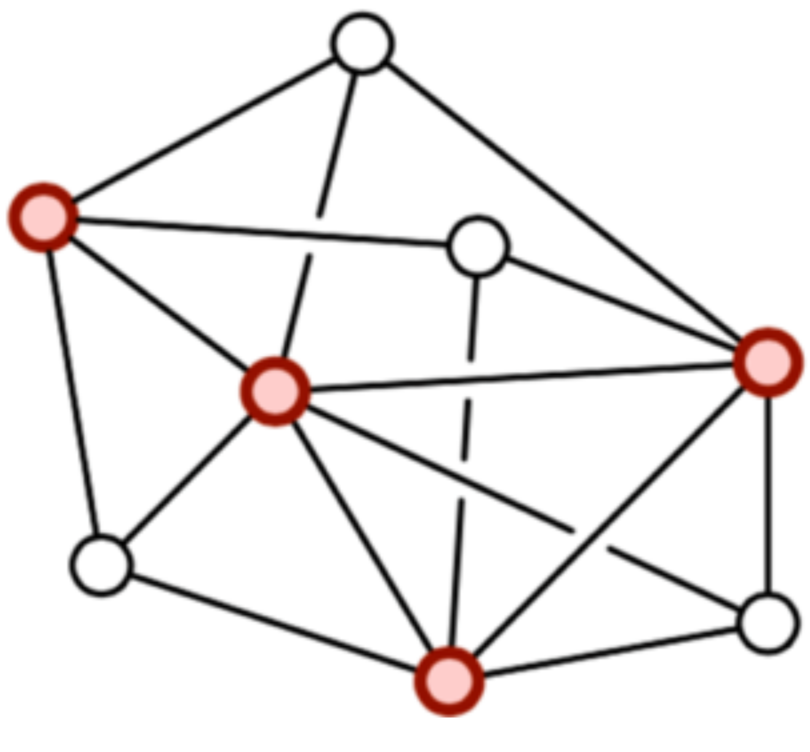
NP hardness reductions II

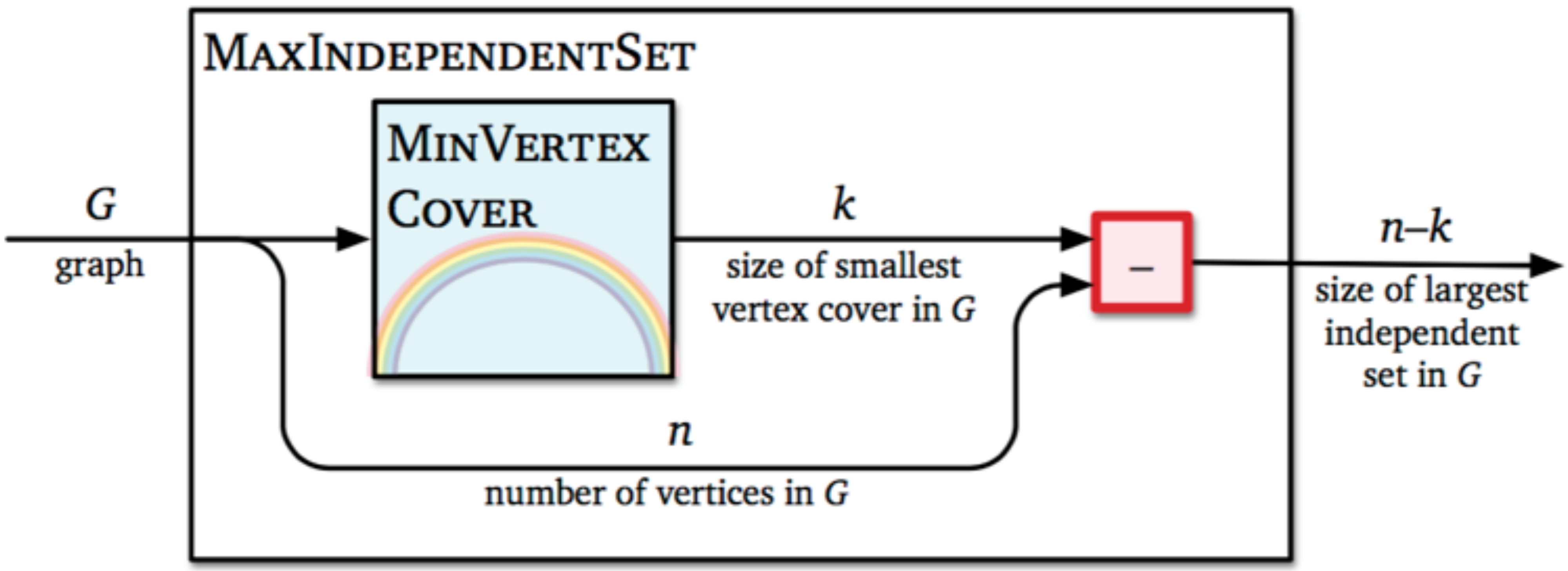
Lecture 23

MIN Vertex Cover

- Input: a graph G(V,E)
- Output: Smallest set of vertices that touch every edge



- If I is Independent set in G, V\I is vertex cover!
- Largest IS in G is the complement of smallest VC in G



what is G'? same graph as G
Output is different

How to prove NP hardness

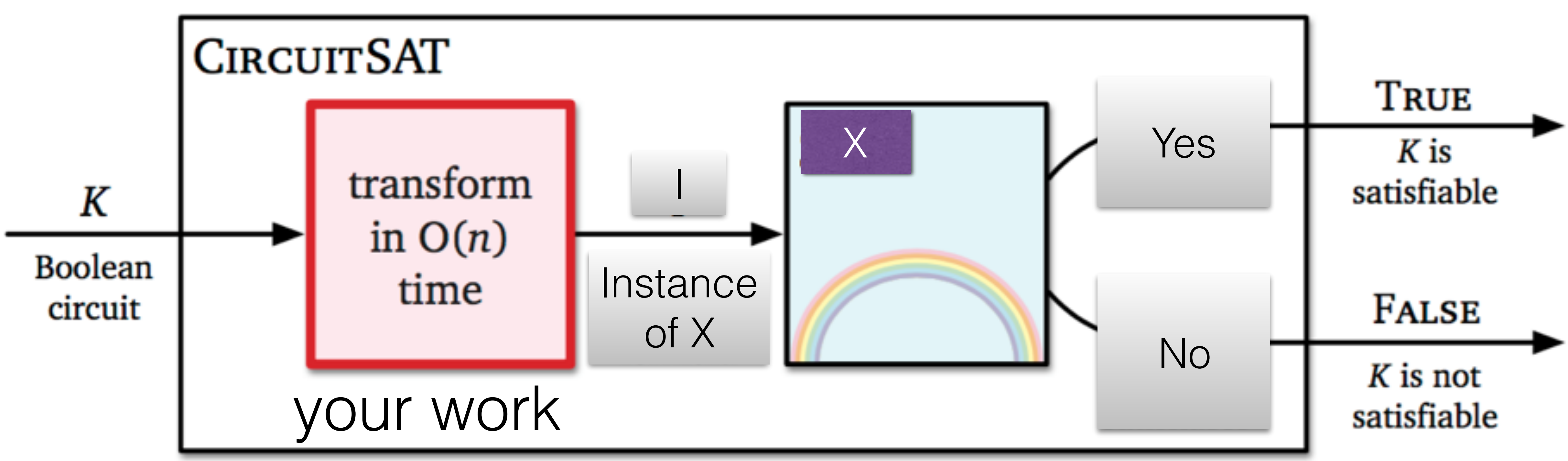
- To prove X is NP-hard:
- **Step 1**: Pick a known NP-hard problem Y
- **Step 2:** Assume for the sake of argument, a polynomial time algorithm for X.
- **Step 3**: Derive a polynomial time algorithm for Y, using algorithm for X as subroutine.
- **Step 4**: Contradiction

Reduce Y to X

Reduce FROM the problem
I know about
TO the problem
I am curious about

NP hardness of X

- To show X is NP hard (example):
- Poly time reduction from CircuitSAT.
- If there is a poly time algorithm to solve X, then there is poly time algorithm to solve CircuitSAT



NP hardness

- Library of NP-hard problems

CircuitSAT

SAT

3SAT

MAX IS

MAX Clique

Min Vertex Cover

SAT

Does a given boolean forumla, in CNF, have a satisfying assignment?

3-SAT

Does a given boolean forumla, in CNF with exactly three literals per clause, have a satisfying assignment?

Min Vertex Cover

In a given undirected graph, what is the (size of the) smallest subset of the vertices covering all of the edges?

Max Independent Set

In a given undirected graph, what is the (size of the) larges subset of the vertices having no edges in common?

Max Clique

What is the (size of the) largest complete subgraph of a given undirected graph?

Min Set Cover

Given a set S and a collection of subsets of S, what is smallest set of these subsets whose union is S?

Min Hitting Set

Given a set S and a collection of subsets of S, what is smallest subset of S containing at least one element from every subset?

Hamilton Path

Does a given graph have a Hamilton Path?

Hamilton Cycle

Does a given graph have a Hamilton Cycle?

Traveling Salesperson

What is the minimum cost Hamilton Cycle in a weighted, complete, graph?

Longest Path

What is the longest path between two given nodes in a weighted, undirected, graph?

Subset Sum

Does a given set of positive integers have a subset with sum k?

Partition

Can a given set of positive integers be partitioned into two subsets each with the same sum?

3-Partition

Can a given set of $3n$ positive integers be partitioned into n 3-element subsets each with the same sum?

Minesweeper

In a given Minesweeper configuration, is it safe to click on a particular square?

Sodoku

Does a given Sodoku puzzle have a solution?

NP hardness

- Library of NP-hard problems

CircuitSAT

SAT

3SAT

MAX IS

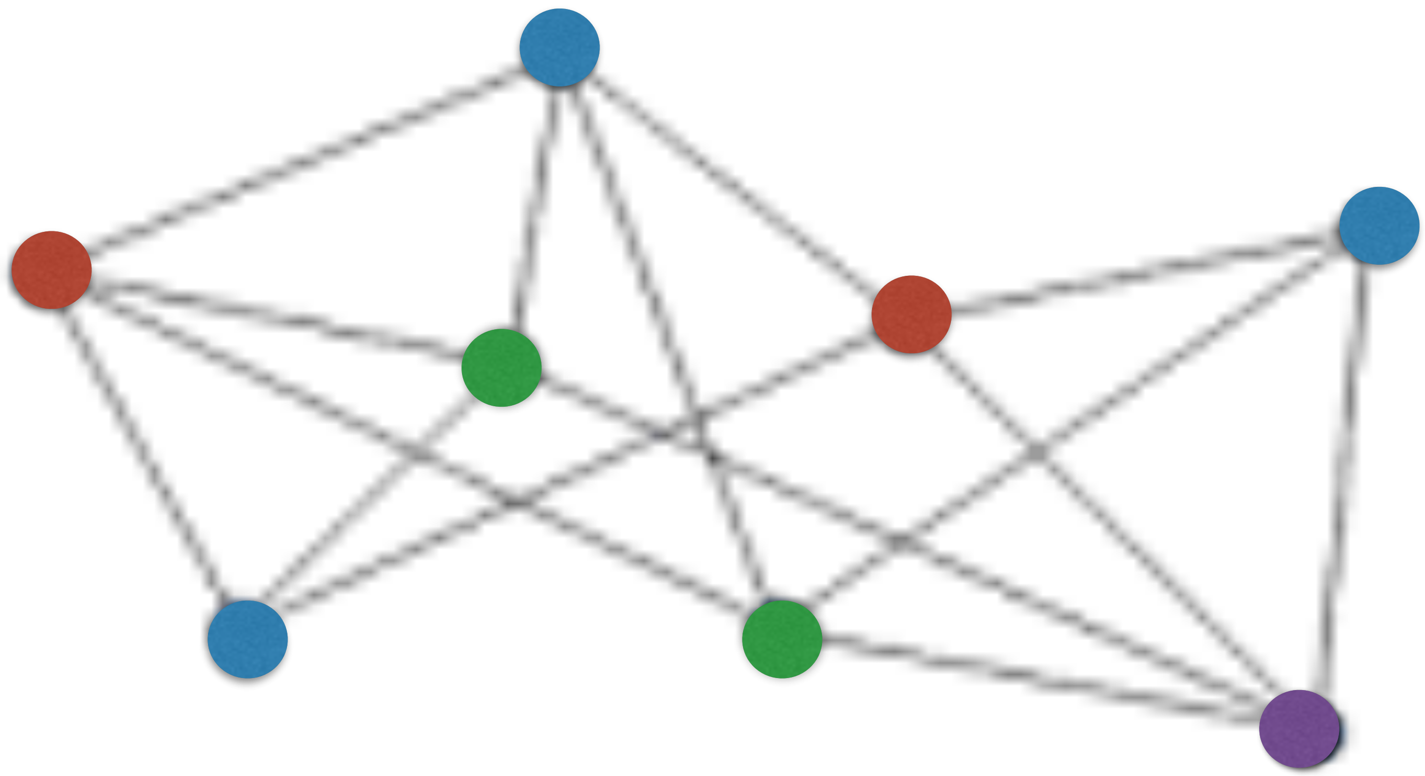
MAX Clique

Min Vertex Cover

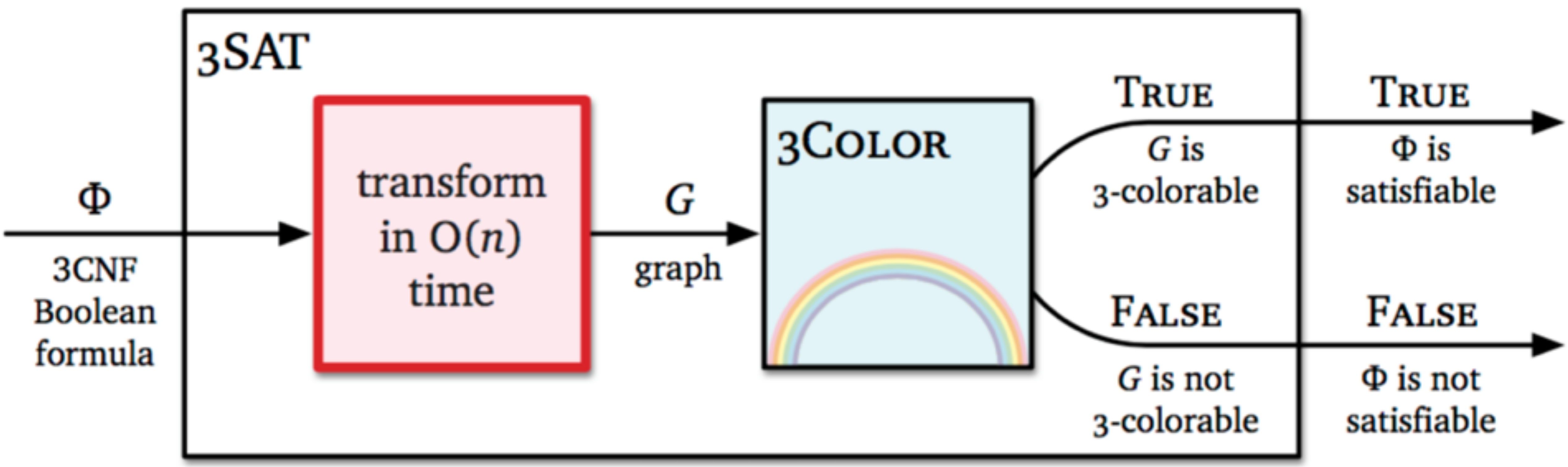
3 Coloring

3 Coloring

- Input: a graph G(V,E)
- Output: True iff G has a proper 3 coloring



what problem to start with?


3SAT
Φ
3CNF
Boolean
formula
transform
in O(n)
time
G
graph
3COLOR
TRUE
G is
3-colorable
FALSE
G is not
3-colorable
TRUE
Φ is
satisfiable
FALSE
Φ is not
satisfiable

3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

Best described in pieces

1) piece that corresponds to variables

2) piece that corresponds to clauses

3) piece that enforces logical consistency

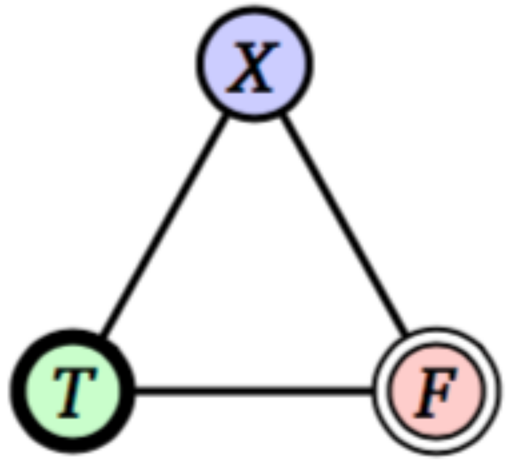
"gadgets"

3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

Best described in pieces

1) Truth Gadget

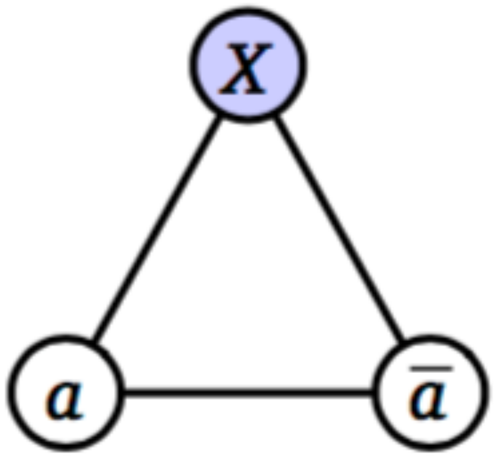


3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

Best described in pieces

2) Variable Gadget



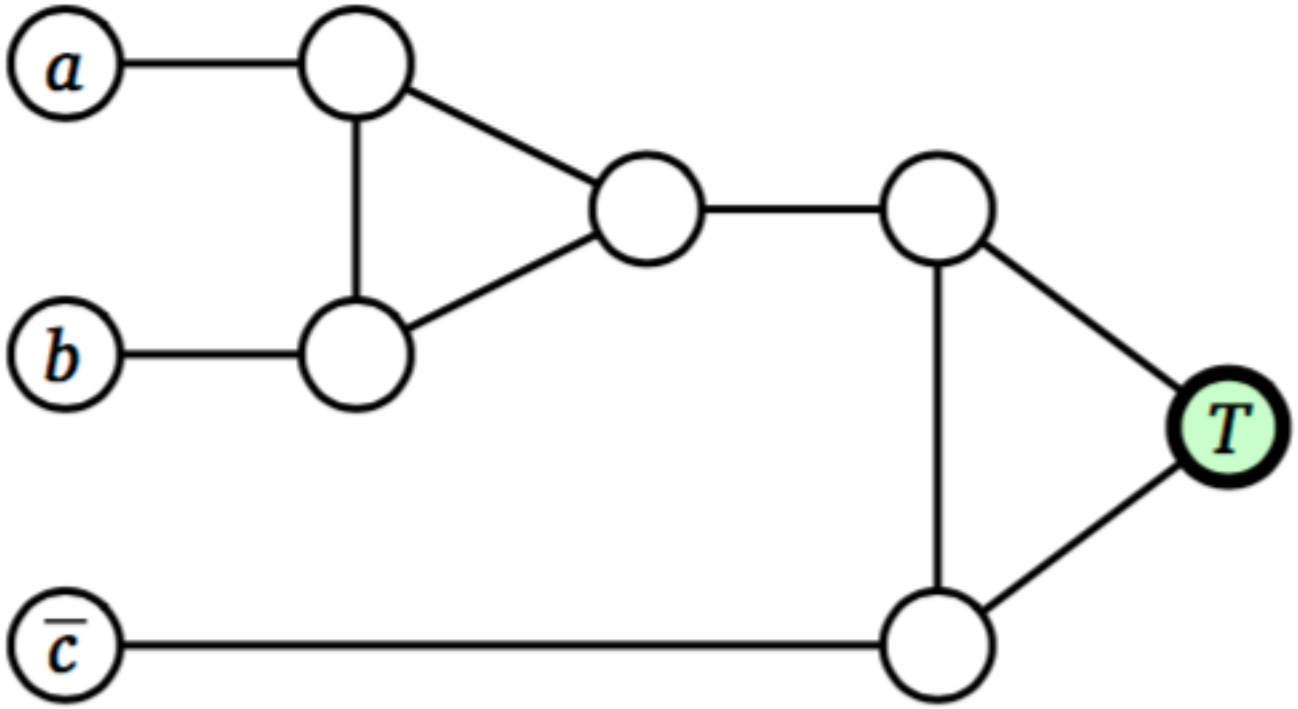
one vertex in the graph for every variable and one for its negation. One vertex labeled X

3COL

- Given an arbitrary 3CNF formula F
- Build a graph G as follows

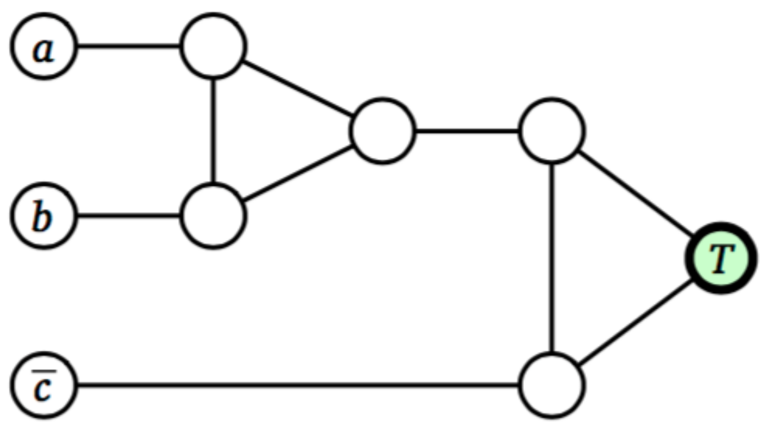
Best described in pieces

3) Clause Gadget



3COL

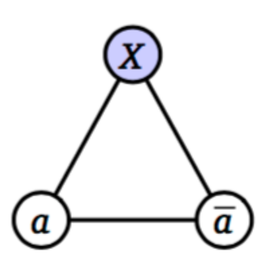
$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$

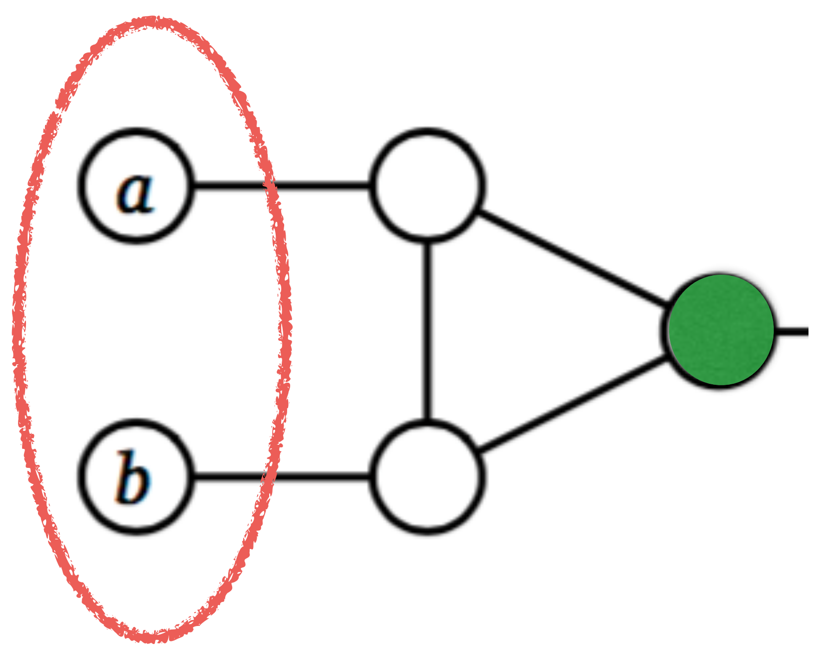


in any proper coloring
at least one of the three literals
must be colored T

easier to prove with 2 SAT example

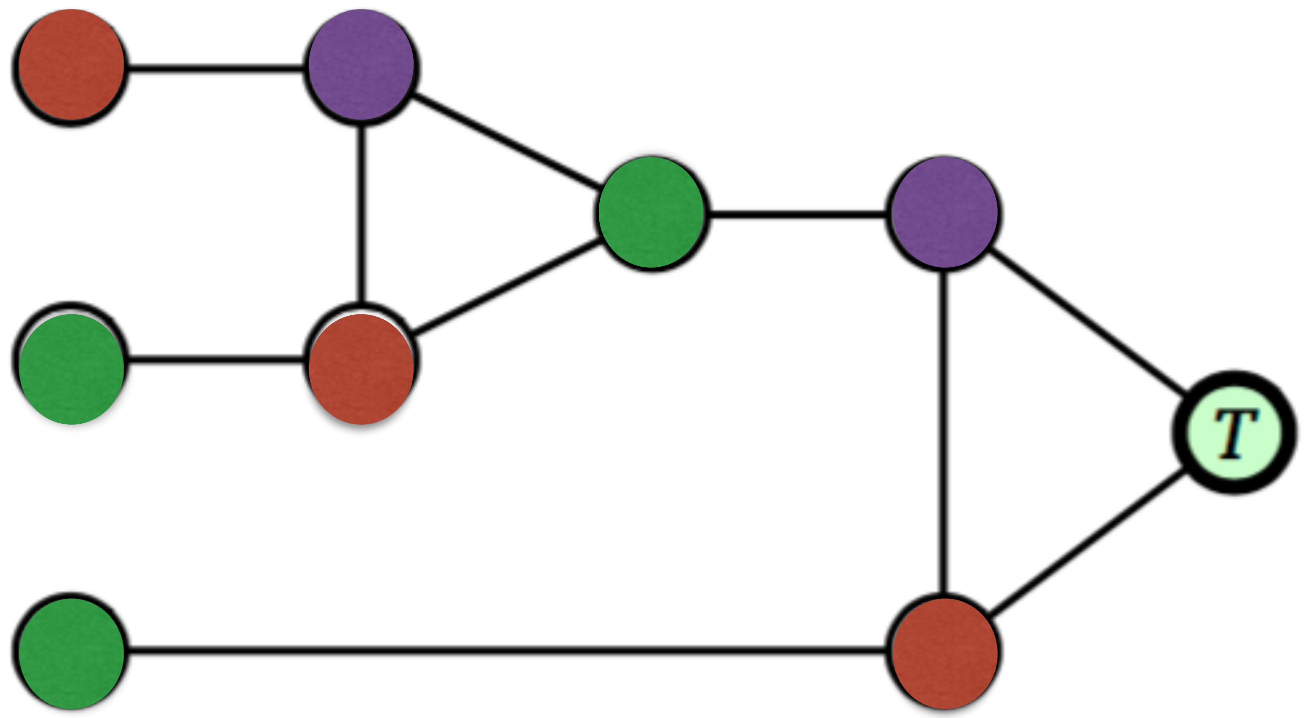
literal vertices,
connected to X





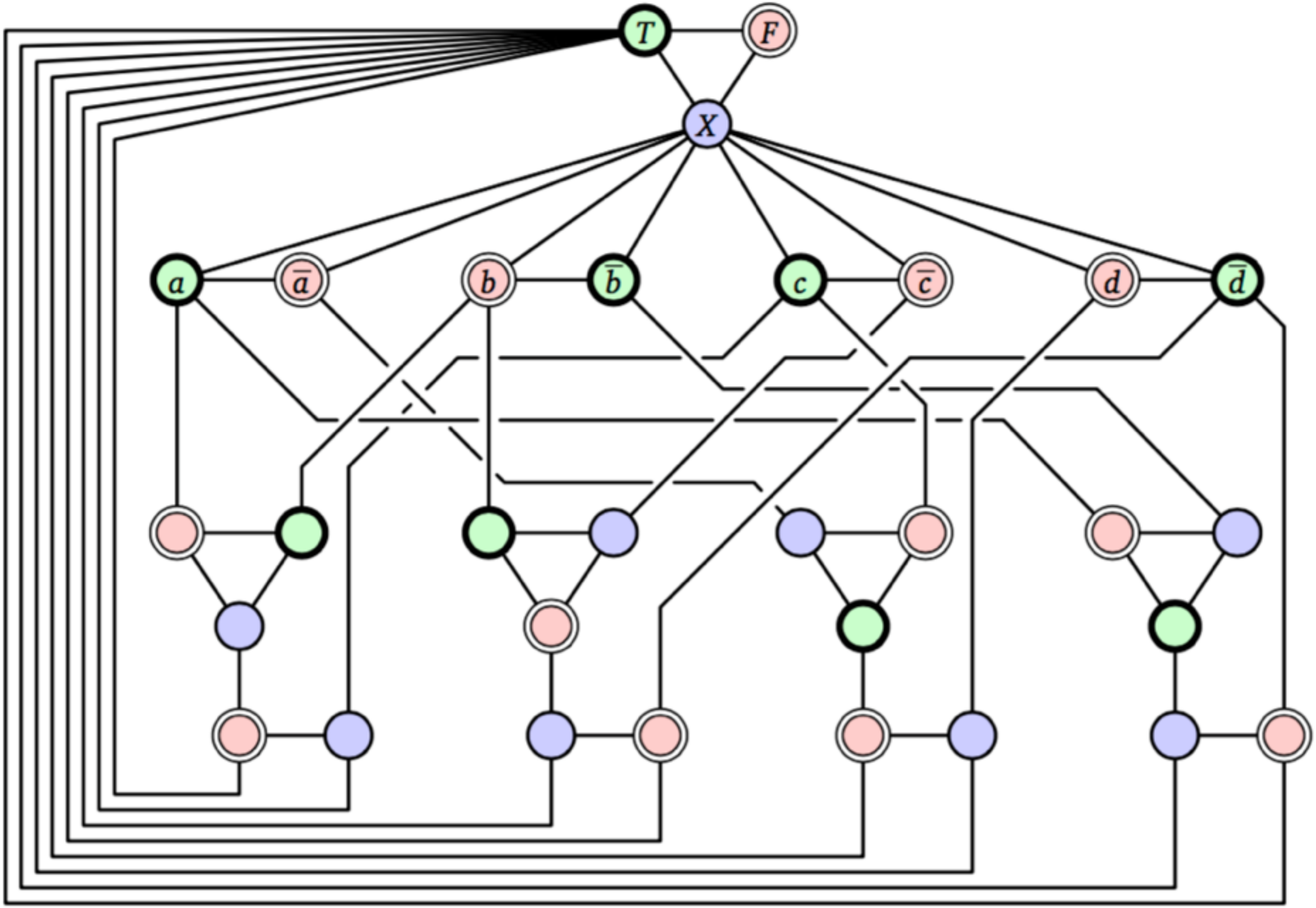
3COL

$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$



There are 8 possible colorings for the 3 literals on the left.

- For 7 of them one gets colored T and I can properly color the gadget
- For the 8th, all of them are colored False and I can't properly color the gadget



A 3-colorable graph derived from the satisfiable 3CNF formula

$$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$$

Proof

Suppose F is satisfiable	Suppose G is 3-Colorable
So G is 3-Colorable	So F is satisfiable

Proof

Suppose F is satisfiable

- Fix any satisfying assignment
- Color True literals same color as T
- Color False literals same color as F
- By case analysis:

extend the coloring to the clause gadget

So G is 3-Colorable

Suppose G is 3-Colorable

So F is satisfiable

Proof

Suppose F is satisfiable

- Fix any satisfying assignment
- Color True literals same color as T
- Color False literals same color as F
- By case analysis:
  extend the coloring to the clause gadget

So G is 3-Colorable

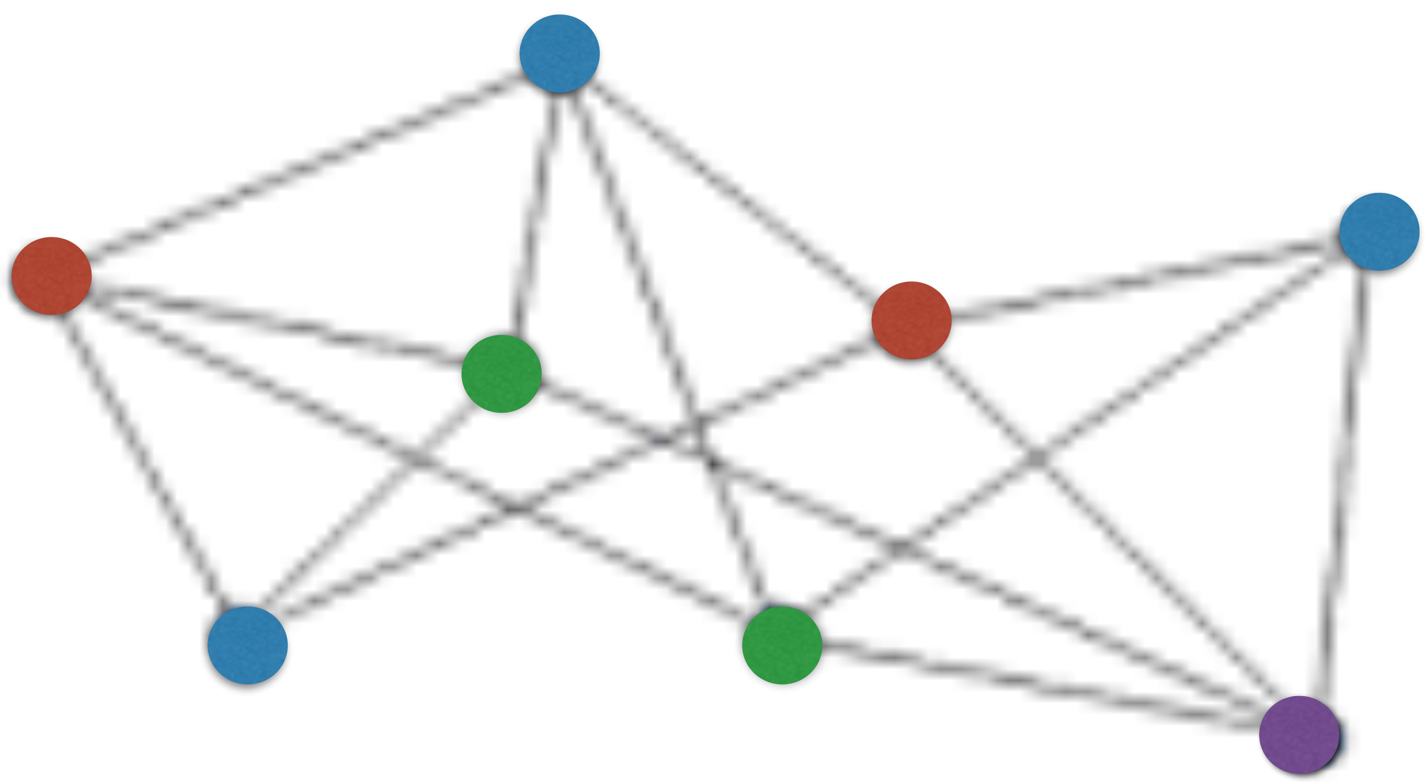
Suppose G is 3-Colorable

- Fix a proper 3 Coloring
- Each literal vertex is colored T or F
- This gives me an assignment of boolean values to variables
- By case analysis: At least one literal in each clause gadget is colored T

So F is satisfiable

4 Coloring?

- Input: a graph G(V,E)
- Output: True iff G has a proper 4 coloring



Hamiltonian Cycle

- Input: a directed graph G(V,E)
- Output: Is there a cycle in G that visits each vertex exactly once?
- Really asking if there is a way to order the vertices so that every adjacent pair is connected by an edge.
- Reduction from HC if a problem asks for ordering of vertices.
- Anti-topological sort

NP hardness

- Library of NP-hard problems

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MAX IS

MAX Clique

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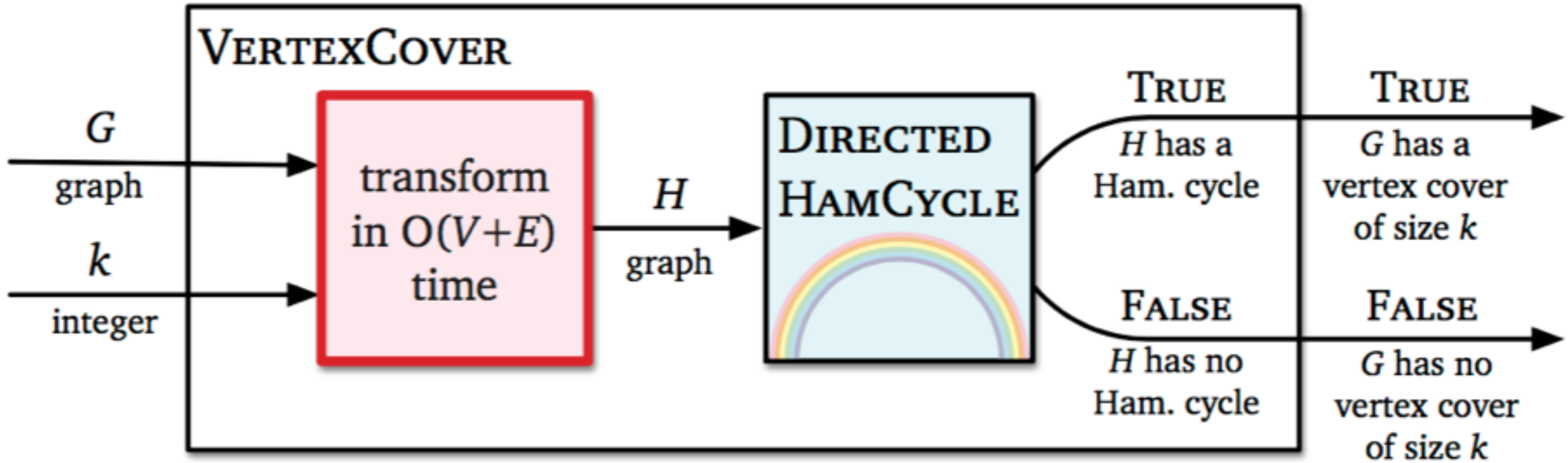
Hamiltonian Cycle



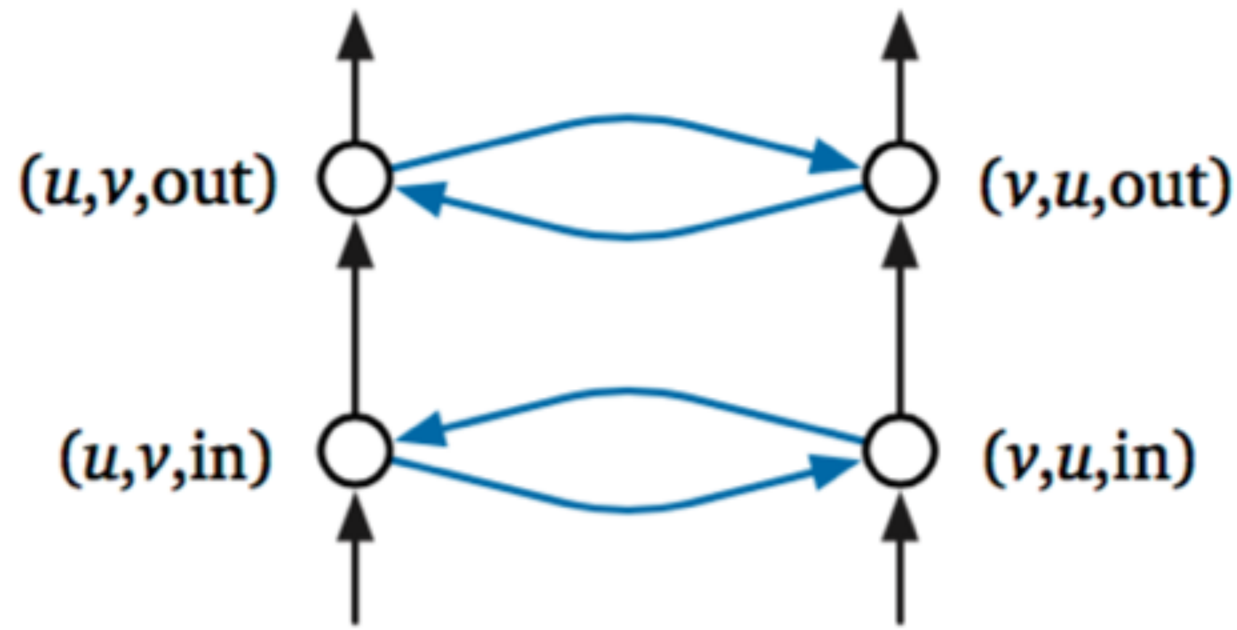
Hamiltonian Cycle

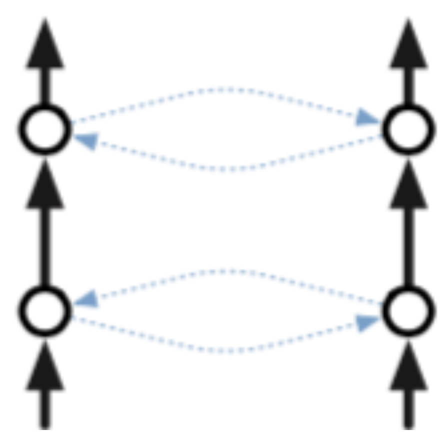
- Given an arbitrary graph G and parameter k
- Build a graph H as follows

Best described in gadgets

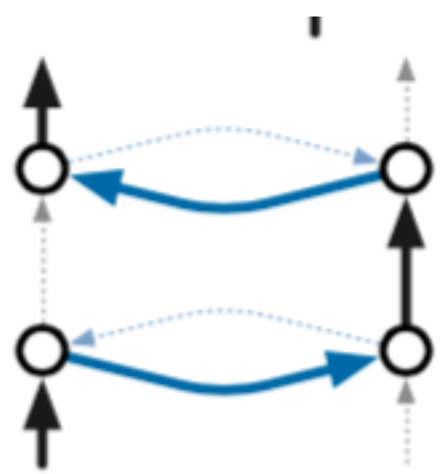
Hamiltonian Cycle

1) edge gadget

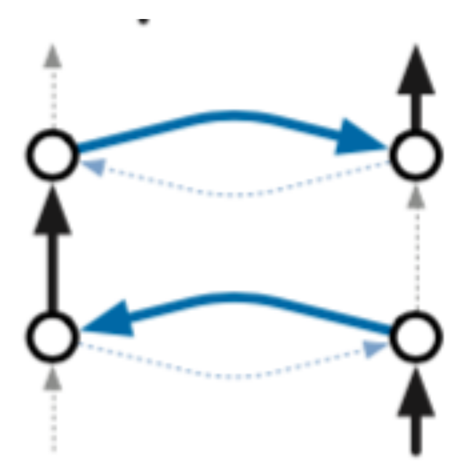




both u,v in VC



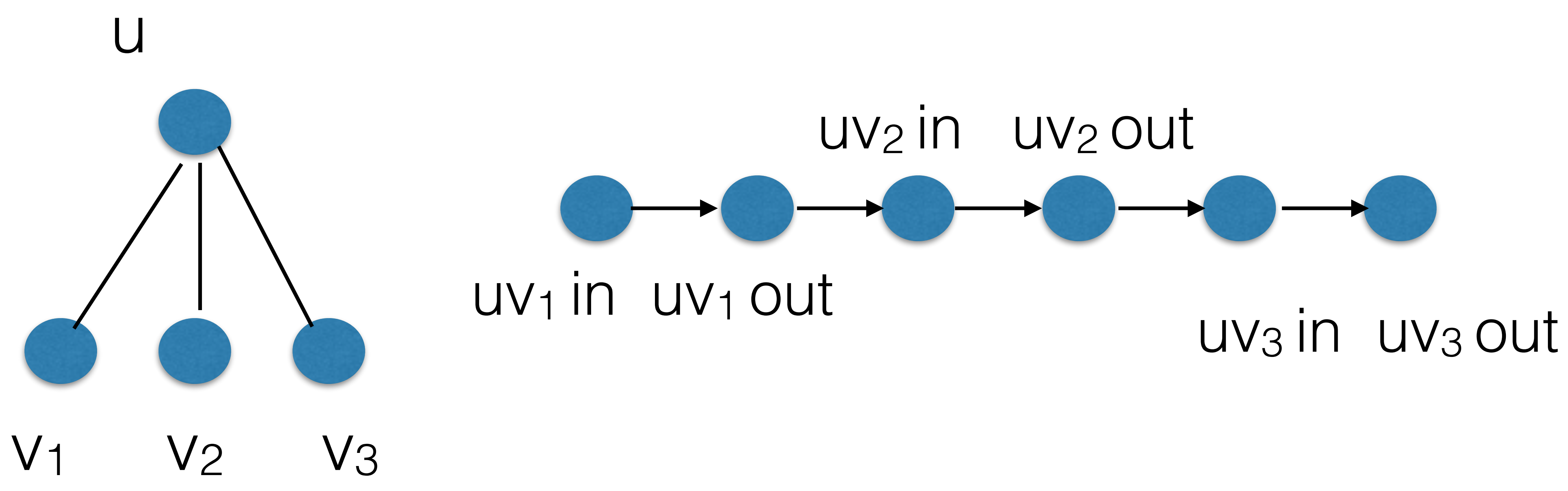
only u in VC



only v in VC

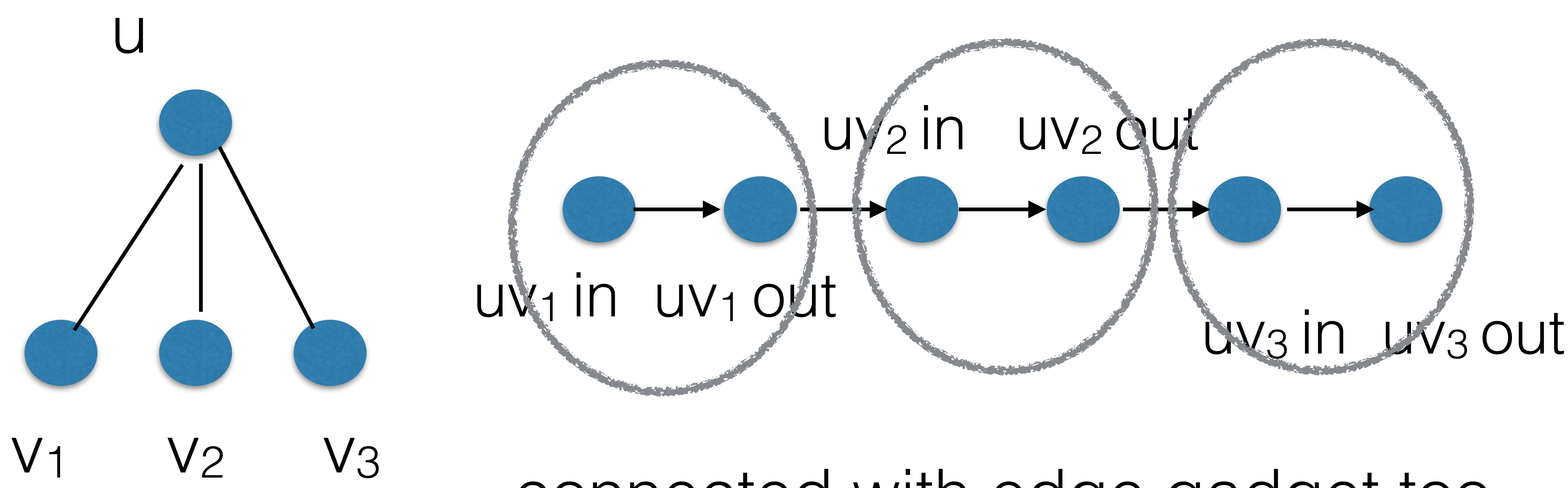
Hamiltonian Cycle

2) vertex gadget



Hamiltonian Cycle

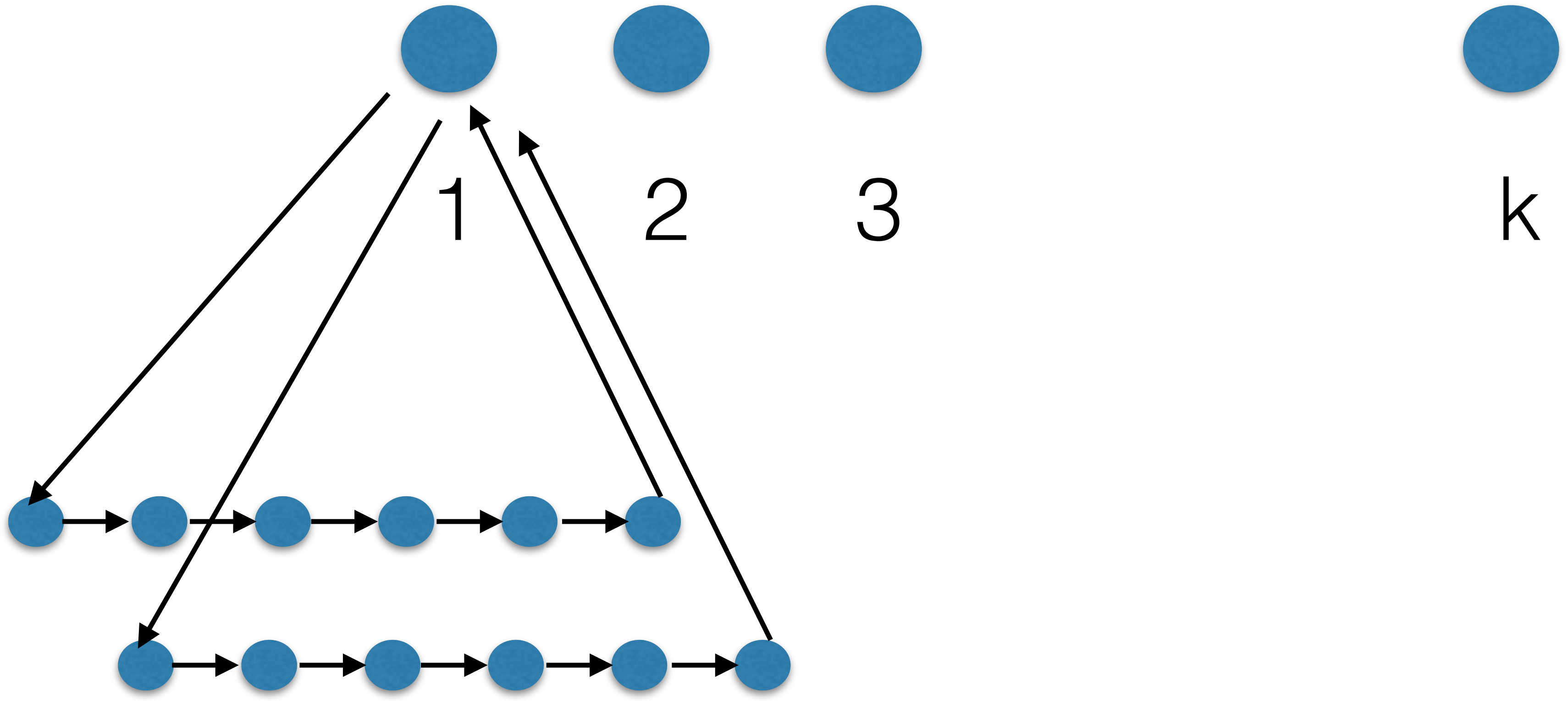
2) vertex gadget

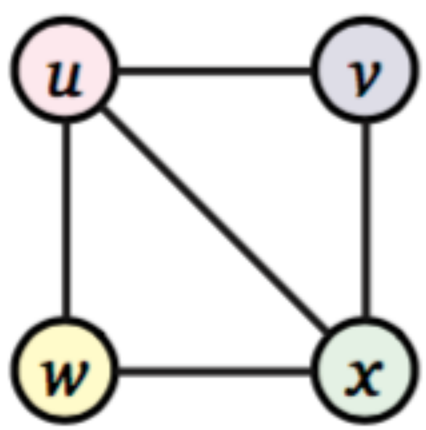


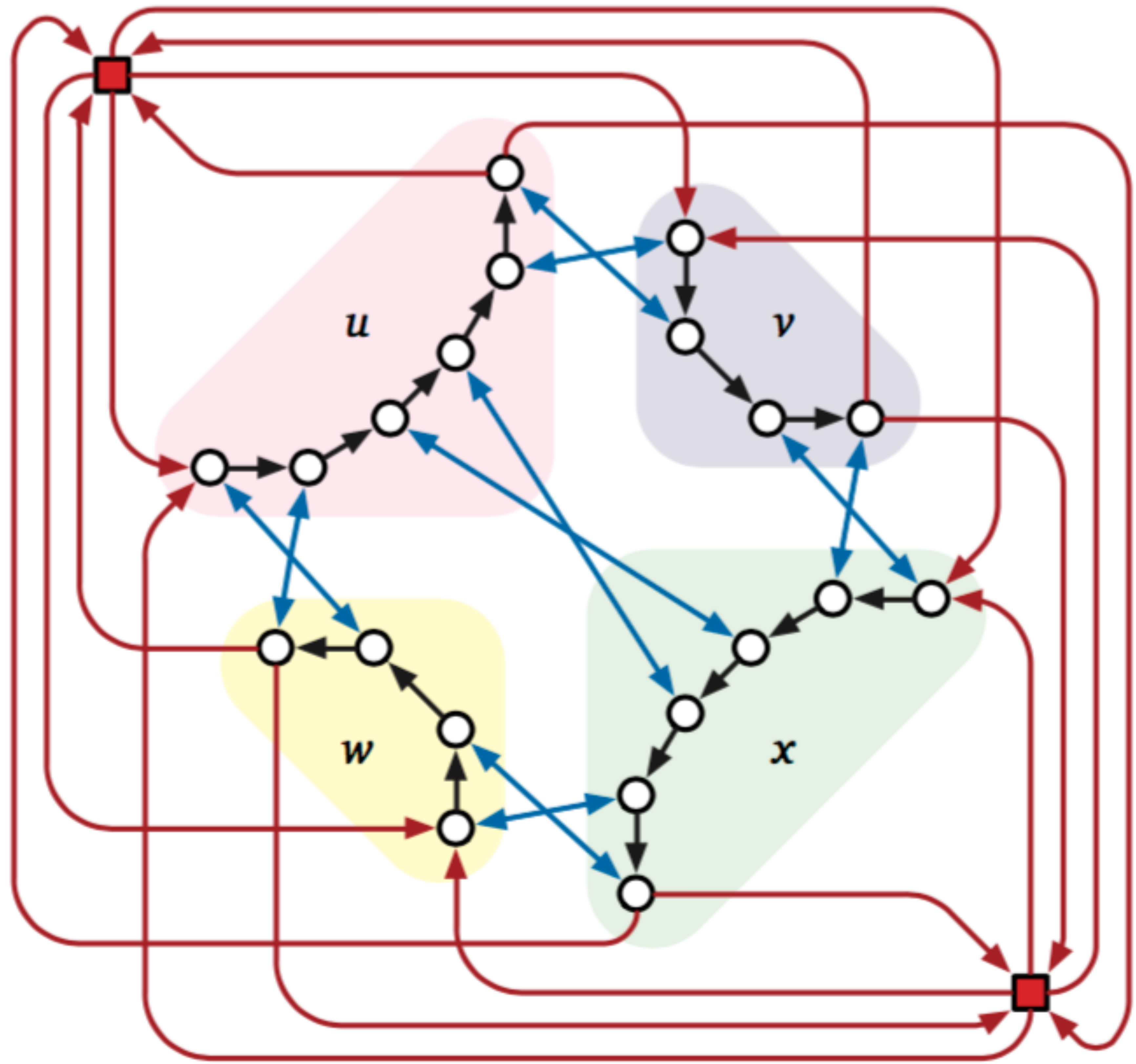
connected with edge gadget too

Hamiltonian Cycle

3) cover gadget




u
v
w
x


u
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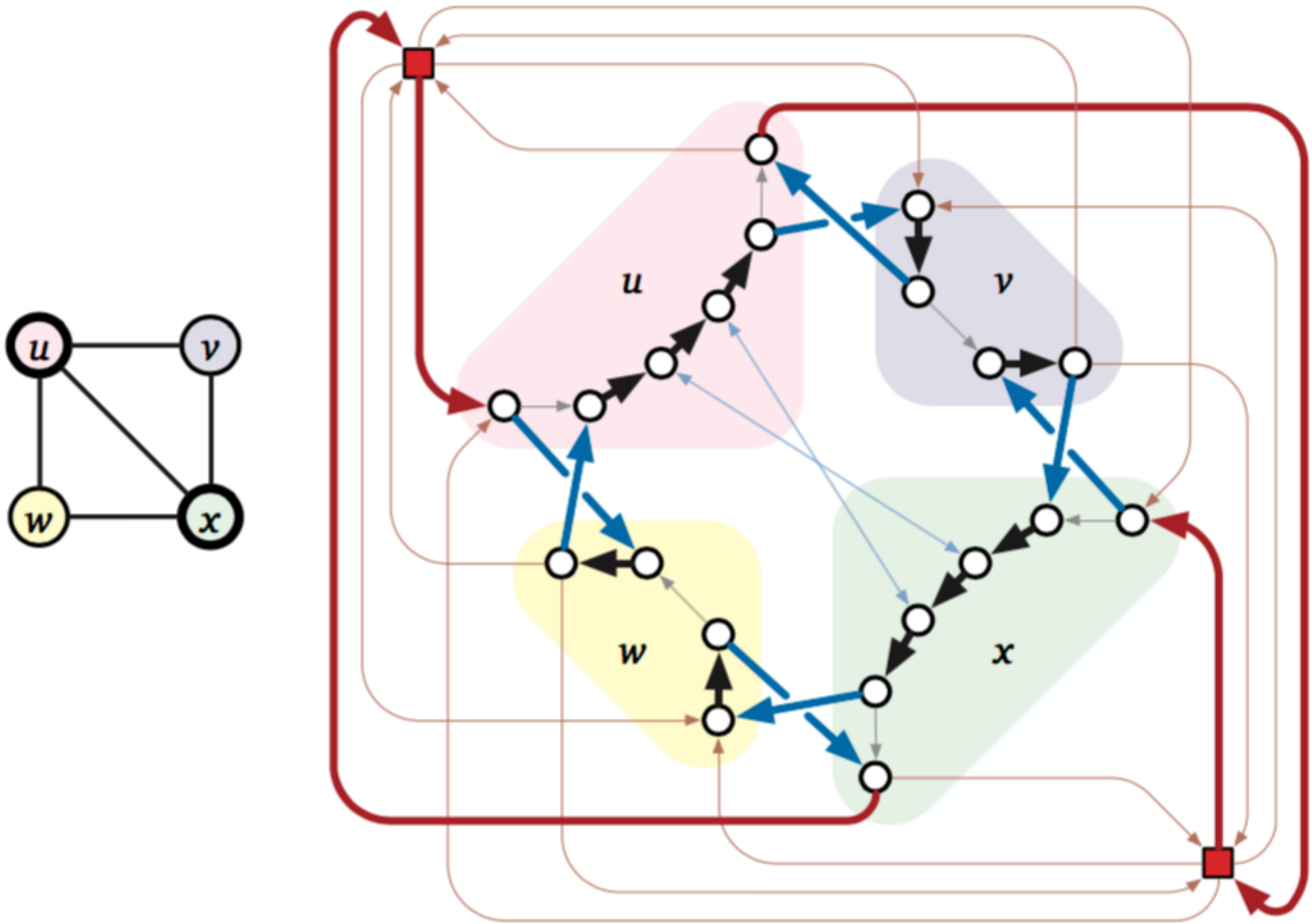

u
v
w
x
u
v
w
x