HW 0: Extra problems

CS/ECE 374: Algorithms & Models of Computation, Fall 2017 Version: **1.0**

Each homework assignment will include at least one solved problem, similar to the problems assigned in that homework, together with the grading rubric we would apply *if* this problem appeared on a homework or exam. These model solutions illustrate our recommendations for structure, presentation, and level of detail in your homework solutions. Of course, the actual *content* of your solutions won't match the model solutions, because your problems are different!

Solved Problems

4. Recall that the *reversal* w^R of a string w is defined recursively as follows:

$$w^R := \begin{cases} \epsilon & \text{if } w = \epsilon \\ x^R \bullet a & \text{if } w = a \cdot x \end{cases}$$

A *palindrome* is any string that is equal to its reversal, like $AMANAPLANACANALPANAMA$, $RACECAR$, $POOP$, I, and the empty string.

(a) Give a recursive definition of a palindrome over the alphabet Σ.

(b) Prove $w = w^R$ for every palindrome w (according to your recursive definition).

(c) Prove that every string w such that $w = w^R$ is a palindrome (according to your recursive definition).

In parts (b) and (c), you may assume without proof that $(x \cdot y)^R = y^R \bullet x^R$ and $(x^R)^R = x$ for all strings x and y.

Solution:

(a) A string $w \in \Sigma^*$ is a palindrome if and only if either

- $w = \epsilon$, or
- $w = a$ for some symbol $a \in \Sigma$, or
- $w = axa$ for some symbol $a \in \Sigma$ and some *palindrome* $x \in \Sigma^*$.

Rubric: 2 points = 1/2 for each base case + 1 for the recursive case. No credit for the rest of the problem unless this is correct.

(b) Let w be an arbitrary palindrome.

Assume that $x = x^R$ for every palindrome x such that $|x| < |w|$.

There are three cases to consider (mirroring the three cases in the definition):

- If $w = \epsilon$, then $w^R = \epsilon$ by definition, so $w = w^R$.
- If $w = a$ for some symbol $a \in \Sigma$, then $w^R = a$ by definition, so $w = w^R$.

- Suppose $w = axa$ for some symbol $a \in \Sigma$ and some palindrome $x \in P$. Then

$$
\begin{aligned}
w^R &= (a \cdot x \bullet a)^R \\
&= (x \bullet a)^R \bullet a && \text{by definition of reversal} \\
&= a^R \bullet x^R \bullet a && \text{You said we could assume this.} \\
&= a \bullet x^R \bullet a && \text{by definition of reversal} \\
&= a \bullet x \bullet a && \text{by the inductive hypothesis} \\
&= w && \text{by assumption}
\end{aligned}
$$

In all three cases, we conclude that $w = w^R$.

Rubric: 4 points: standard induction rubric (scaled)

(c) Let w be an arbitrary string such that $w = w^R$.

Assume that every string x such that $|x| < |w|$ and $x = x^R$ is a palindrome.

There are three cases to consider (mirroring the definition of "palindrome"):

- If $w = \epsilon$, then w is a palindrome by definition.
- If $w = a$ for some symbol $a \in \Sigma$, then w is a palindrome by definition.
- Otherwise, we have $w = ax$ for some symbol a and some *non-empty* string x.
  The definition of reversal implies that $w^R = (ax)^R = x^R a$.
  Because x is non-empty, its reversal x^R is also non-empty.
  Thus, $x^R = by$ for some symbol b and some string y.
  It follows that $w^R = bya$, and therefore $w = (w^R)^R = (bya)^R = ay^R b$.

  [At this point, we need to prove that $a = b$ and that y is a palindrome.]

  Our assumption that $w = w^R$ implies that $bya = ay^R b$.
  The recursive definition of string equality immediately implies $a = b$.

  Because $a = b$, we have $w = ay^R a$ and $w^R = aya$.
  The recursive definition of string equality implies $y^R a = ya$.
  It immediately follows that $(y^R a)^R = (ya)^R$.
  Known properties of reversal imply $(y^R a)^R = a(y^R)^R = ay$ and $(ya)^R = ay^R$.
  It follows that $ay^R = ay$, and therefore $y = y^R$.
  The inductive hypothesis now implies that y is a palindrome.

  We conclude that w is a palindrome by definition.

In all three cases, we conclude that w is a palindrome.

Rubric: 4 points: standard induction rubric (scaled).

- No penalty for jumping from $aya = ay^R a$ directly to $y = y^R$.

Rubric:[induction] For problems worth 10 points:

+ 1 for explicitly considering an *arbitrary* object
+ 2 for a valid **strong** induction hypothesis
  - **Deadly Sin!** Automatic zero for stating a weak induction hypothesis, unless the rest of the proof is *perfect*.
+ 2 for explicit exhaustive case analysis

  - No credit here if the case analysis omits an infinite number of objects. (For example: all odd-length palindromes.)
  - −1 if the case analysis omits an finite number of objects. (For example: the empty string.)
  - −1 for making the reader infer the case conditions. Spell them out!
  - No penalty if cases overlap (for example:
- \+ 1 for cases that do not invoke the inductive hypothesis ("base cases")
  - No credit here if one or more "base cases" are missing.
- \+ 2 for correctly applying the *stated* inductive hypothesis
  - No credit here for applying a *different* inductive hypothesis, even if that different inductive hypothesis would be valid.
- \+ 2 for other details in cases that invoke the inductive hypothesis ("inductive cases")
  - No credit here if one or more "inductive cases" are missing.