# Context Free Languages and Grammars

Lecture 7

Tuesday, September 19, 2017

# What stack got to do with it?

## What's a stack but a second hand memory?

1. DFA/NFA/Regular expressions.
   $\equiv$ constant memory computation.
2. Turing machines DFA/NFA + unbounded memory.
   $\equiv$ a standard computer/program.

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3. Turing machines DFA/NFA $+$ unbounded memory.
   $\equiv$ a standard computer/program.
   $\equiv$ NFA with two stacks.

# Context Free Languages and Grammars

- Programming Language Specification
- Parsing
- Natural language understanding
- Generative model giving structure
- ...

# Programming Languages

```
<relational-expression> ::= <shift-expression>
                          | <relational-expression> < <shift-expression>
                          | <relational-expression> > <shift-expression>
                          | <relational-expression> <= <shift-expression>
                          | <relational-expression> >= <shift-expression>

<shift-expression> ::= <additive-expression>
                     | <shift-expression> << <additive-expression>
                     | <shift-expression> >> <additive-expression>

<additive-expression> ::= <multiplicative-expression>
                        | <additive-expression> + <multiplicative-expression>
                        | <additive-expression> - <multiplicative-expression>

<multiplicative-expression> ::= <cast-expression>
                              | <multiplicative-expression> * <cast-expression>
                              | <multiplicative-expression> / <cast-expression>
                              | <multiplicative-expression> % <cast-expression>

<cast-expression> ::= <unary-expression>
                    | ( <type-name> ) <cast-expression>

<unary-expression> ::= <postfix-expression>
                     | ++ <unary-expression>
                     | -- <unary-expression>
                     | <unary-operator> <cast-expression>
                     | sizeof <unary-expression>
                     | sizeof <type-name>

<postfix-expression> ::= <primary-expression>
                       | <postfix-expression> [ <expression> ]
                       | <postfix-expression> ( {<assignment-expression>}* )
                       | <postfix-expression> . <identifier>
                       | <postfix-expression> -> <identifier>
                       | <postfix-expression> ++
                       | <postfix-expression> --
```

# Natural Language Processing

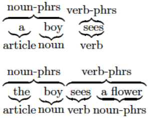
English sentences can be described as

$$
\begin{aligned}
&\langle S\rangle \rightarrow \langle NP\rangle\langle VP\rangle\\
&\langle NP\rangle \rightarrow \langle CN\rangle \mid \langle CN\rangle\langle PP\rangle\\
&\langle VP\rangle \rightarrow \langle CV\rangle \mid \langle CV\rangle\langle PP\rangle\\
&\langle PP\rangle \rightarrow \langle P\rangle\langle CN\rangle\\
&\langle CN\rangle \rightarrow \langle A\rangle\langle N\rangle\\
&\langle CV\rangle \rightarrow \langle V\rangle \mid \langle V\rangle\langle NP\rangle\\
&\langle A\rangle \rightarrow \text{a} \mid \text{the}\\
&\langle N\rangle \rightarrow \text{boy} \mid \text{girl} \mid \text{flower}\\
&\langle V\rangle \rightarrow \text{touches} \mid \text{likes} \mid \text{sees}\\
&\langle P\rangle \rightarrow \text{with}
\end{aligned}
$$

---

**English Sentences**

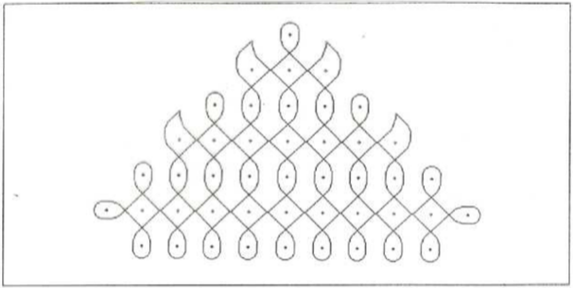
*Examples*



# Models of Growth

- ***L***-systems
- http://www.kevs3d.co.uk/dev/lsystems/

# Kolam drawing generated by grammar



# Context Free Grammar (CFG) Definition

## Definition

A CFG is a quadruple $G = (V, T, P, S)$

- $V$ is a finite set of non-terminal symbols
- $T$ is a finite set of terminal symbols (alphabet)
- $P$ is a finite set of productions, each of the form
  $A \rightarrow \alpha$
  where $A \in V$ and $\alpha$ is a string in $(V \cup T)^*$.
  Formally, $P \subset V \times (V \cup T)^*$.
- $S \in V$ is a start symbol

$G = ($ Variables, Terminals, Productions, Start var $)$

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# Example

- $V = \{S\}$
- $T = \{a, b\}$
- $P = \{S \rightarrow \epsilon \mid a \mid b \mid aSa \mid bSb\}$
  (abbrev. for $S \rightarrow \epsilon, S \rightarrow a, S \rightarrow b, S \rightarrow aSa, S \rightarrow bSb$)

$$S \rightsquigarrow aSa \rightsquigarrow abSba \rightsquigarrow abbSbba \rightsquigarrow abb\,b\,bba$$

What strings can $S$ generate like this?

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# Example formally...

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$$G = \left( \{S\}, \quad \{a, b\}, \quad \left\{ \begin{array}{c} S \to \epsilon, \\ S \to a, \\ S \to b \\ S \to aSa \\ S \to bSb \end{array} \right\} \quad S \right)$$

# Palindromes

- Madam in Eden I'm Adam
- Dog doo? Good God!
- Dogma: I am God.
- A man, a plan, a canal, Panama
- Are we not drawn onward, we few, drawn onward to new era?
- Doc, note: I dissent. A fast never prevents a fatness. I diet on cod.
- http://www.palindromelist.net

# Examples

$L = \{0^n 1^n \mid n \geq 0\}$

$S \rightarrow \epsilon \mid 0S1$

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# Notation and Convention

Let $G = (V, T, P, S)$ then

- $a, b, c, d, \ldots,$ in $T$ (terminals)
- $A, B, C, D, \ldots,$ in $V$ (non-terminals)
- $u, v, w, x, y, \ldots$ in $T^*$ for strings of terminals
- $\alpha, \beta, \gamma, \ldots$ in $(V \cup T)^*$
- $X, Y, X$ in $V \cup T$

# "Derives" relation

Formalism for how strings are derived/generated

## Definition

Let $G = (V, T, P, S)$ be a CFG. For strings $\alpha_1, \alpha_2 \in (V \cup T)^*$ we say $\alpha_1$ derives $\alpha_2$ denoted by $\alpha_1 \rightsquigarrow_G \alpha_2$ if there exist strings $\beta, \gamma, \delta$ in $(V \cup T)^*$ such that

- $\alpha_1 = \beta A \delta$
- $\alpha_2 = \beta \gamma \delta$
- $A \rightarrow \gamma$ is in $P$.

**Examples:** $S \rightsquigarrow \epsilon$, $S \rightsquigarrow 0S1$, $0S1 \rightsquigarrow 00S11$, $0S1 \rightsquigarrow 01$.

# "Derives" relation continued

## Definition

For integer $k \geq 0$, $\alpha_1 \leadsto^k \alpha_2$ inductive defined:

- $\alpha_1 \leadsto^0 \alpha_2$ if $\alpha_1 = \alpha_2$
- $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto \beta_1$ and $\beta_1 \leadsto^{k-1} \alpha_2$.
- Alternative definition: $\alpha_1 \leadsto^k \alpha_2$ if $\alpha_1 \leadsto^{k-1} \beta_1$ and $\beta_1 \leadsto \alpha_2$

$\leadsto^*$ is the reflexive and transitive closure of $\leadsto$.

$\alpha_1 \leadsto^* \alpha_2$ if $\alpha_1 \leadsto^k \alpha_2$ for some $k$.

**Examples:** $S \leadsto^* \epsilon$, $0S1 \leadsto^* 0000011111$.

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The language generated by CFG $G = (V, T, P, S)$ is denoted by $L(G)$ where $L(G) = \{w \in T^* \mid S \rightsquigarrow^* w\}$.

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$S \to \epsilon \mid 0S1$

$L = \{0^n 1^m \mid m > n\}$

$L = \left\{ w \in \{(,)\}^* \;\middle|\; w \text{ is properly nested string of parenthesis} \right\}$

# Closure Properties of CFLs

$G_1 = (V_1, T, P_1, S_1)$ and $G_2 = (V_2, T, P_2, S_2)$

**Assumption:** $V_1 \cap V_2 = \emptyset$, that is, non-terminals are not shared

**Theorem**

CFLs *are closed under union.* $L_1, L_2$ CFLs *implies* $L_1 \cup L_2$ *is a* CFL.

**Theorem**

CFLs *are closed under concatenation.* $L_1, L_2$ CFLs *implies* $L_1 \bullet L_2$ *is a* CFL.

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*If* $L$ *is a* CFL $\implies$ $L^*$ *is a* CFL.

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## Concatenation

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# Closure Properties of CFLs

## Stardom (i.e, Kleene star)

**Theorem**

*CFLs are closed under Kleene star.*
*If $L$ is a CFL $\implies$ $L^*$ is a CFL.*

# Exercise

- Prove that every regular language is context-free using previous closure properties.
- Prove the set of regular expressions over an alphabet $\Sigma$ forms a non-regular language which is context-free.

# Closure Properties of CFLs continued

**Theorem**

*CFLs are **not** closed under complement or intersection.*

**Theorem**

*If $L_1$ is a CFL and $L_2$ is regular then $L_1 \cap L_2$ is a CFL.*

# Canonical non-CFL

**Theorem**

$L = \{a^n b^n c^n \mid n \geq 0\}$ *is not context-free.*

Proof based on pumping lemma for CFLs. Technical and outside the scope of this class.

# Parse Trees or Derivation Trees

A tree to represent the derivation $S \rightsquigarrow^* w$.

- Rooted tree with root labeled $S$
- Non-terminals at each internal node of tree
- Terminals at leaves
- Children of internal node indicate how non-terminal was expanded using a production rule

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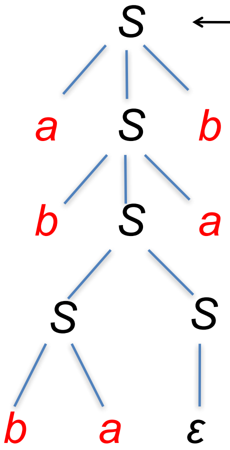



A derivation tree for *abbaab*

(also called "parse tree")

$S \to aSb \mid bSa \mid SS \mid ab \mid ba \mid \varepsilon$

*A* corresponding derivation of *abbaab*

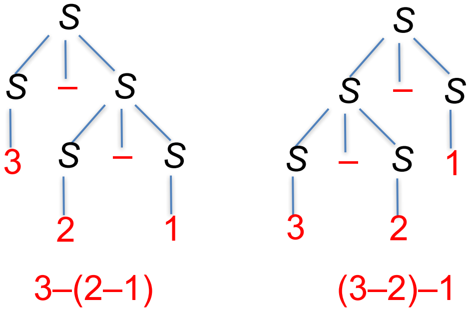
$S \to aSb \to abSab \to abSSab \to abbaSab \to abbaab$

# Ambiguity in CFLs

## Definition

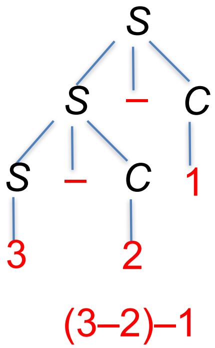
A CFG $G$ is ambiguous if there is a string $w \in L(G)$ with two different parse trees. If there is no such string then $G$ is unambiguous.

**Example:** $S \to S - S \mid 1 \mid 2 \mid 3$



# Ambiguity in CFLs

- Original grammar: $S \to S - S \mid 1 \mid 2 \mid 3$
- Unambiguous grammar:
  $S \to S - C \mid 1 \mid 2 \mid 3$
  $C \to 1 \mid 2 \mid 3$




The grammar forces a parse corresponding to left-to-right evaluation.

# Inherently ambiguous languages

## Definition

A CFL $L$ is inherently ambiguous if there is no unambiguous CFG $G$ such that $L = L(G)$.

- There exist inherently ambiguous CFLs.
  **Example:** $L = \{a^n b^m c^k \mid n = m \text{ or } m = k\}$
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# Inductive proofs for CFGs

**Question:** How do we formally prove that a CFG $L(G) = L$?

**Example:** $S \to \epsilon \mid a \mid b \mid aSa \mid bSb$

**Theorem**

$L(G) = \{palindromes\} = \{w \mid w = w^R\}$

Two directions:

- $L(G) \subseteq L$, that is, $S \rightsquigarrow^* w$ then $w = w^R$
- $L \subseteq L(G)$, that is, $w = w^R$ then $S \rightsquigarrow^* w$

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# $L(G) \subseteq L$

Show that if $S \rightsquigarrow^* w$ then $w = w^R$

By induction on length of derivation, meaning
For all $k \geq 1$, $S \rightsquigarrow^{*k} w$ implies $w = w^R$.

- If $S \rightsquigarrow^1 w$ then $w = \epsilon$ or $w = a$ or $w = b$. Each case $w = w^R$.
- Assume that for all $k < n$, that if $S \rightarrow^k w$ then $w = w^R$
- Let $S \rightsquigarrow^n w$ (with $n > 1$). Wlog $w$ begin with $a$.
  - Then $S \rightarrow aSa \rightsquigarrow^{k-1} aua$ where $w = aua$.
  - And $S \rightsquigarrow^{n-1} u$ and hence IH, $u = u^R$.
  - Therefore $w^r = (aua)^R = (ua)^R a = au^R a = aua = w$.

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# $L \subseteq L(G)$

Show that if $w = w^R$ then $S \rightsquigarrow^* w$.

By induction on $|w|$
That is, for all $k \geq 0$, $|w| = k$ and $w = w^R$ implies $S \rightsquigarrow^* w$.

**Exercise:** Fill in proof.

# Mutual Induction

Situation is more complicated with grammars that have multiple non-terminals.

See Section 5.3.2 of the notes for an example proof.

# Normal Forms

Normal forms are a way to restrict form of production rules

**Advantage:** Simpler/more convenient algorithms and proofs

Two standard normal forms for CFGs

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- Greibach normal form

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Chomsky Normal Form:

- Productions are all of the form $A \rightarrow BC$ or $A \rightarrow a$. If $\epsilon \in L$ then $S \rightarrow \epsilon$ is also allowed.
- Every CFG $G$ can be converted into CNF form via an efficient algorithm
- Advantage: parse tree of constant degree.

Greibach Normal Form:

- Only productions of the form $A \rightarrow a\beta$ are allowed.
- All CFLs without $\epsilon$ have a grammar in GNF. Efficient algorithm.
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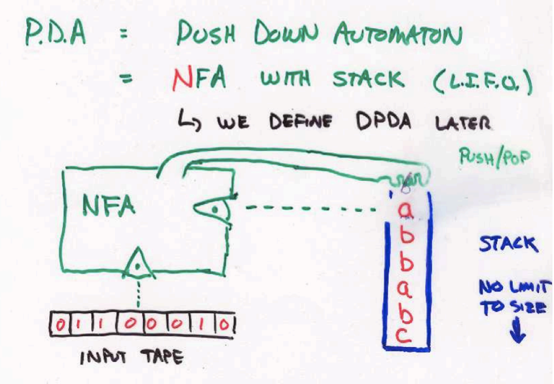
# Things to know: Pushdown Automata

PDA: a NFA coupled with a stack




PDAs and CFGs are equivalent: both generate exactly CFLs.
PDA is a machine-centric view of CFLs.