Polynomial Time Reductions

Lecture 22

Tuesday, November 28, 2017

Part I

(Polynomial Time) Reductions

Reductions

Reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

Using Reductions

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Reduction from Problem $\boldsymbol{X}$ to Problem $\boldsymbol{Y}$ means (informally) that if we have an algorithm for Problem $\boldsymbol{Y}$, we can use it to find an algorithm for Problem $\boldsymbol{X}$.

Using Reductions

1. We use reductions to find algorithms to solve problems.
2. We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)

Reductions for decision problems/languages

For languages L_X, L_Y, a ***reduction from L_X to L_Y*** is:

1. An algorithm ...
2. Input: $w \in \Sigma^*$
3. Output: $w' \in \Sigma^*$
4. Such that:

$$\boxed{w \in L_Y} \iff \boxed{w' \in L_X}$$

(Actually, this is only one type of reduction, but this is the one we'll use most often.) There are other kinds of reductions.

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Reductions for decision problems/languages

For decision problems X, Y, a ***reduction from X to Y*** is:

1. An algorithm ...
2. Input: I_X, an instance of X.
3. Output: I_Y an instance of Y.
4. Such that:
   $$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$$

Using reductions to solve problems

1. $\mathcal{R}$: Reduction $X \to Y$
2. $\mathcal{A}_Y$: algorithm for Y:
3. $\implies$ New algorithm for X:

```
A_X(I_X):
        // I_X: instance of X.
        I_Y <= R(I_X)
        return A_Y(I_Y)
```

If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

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$\mathcal{A}_X(I_X)$:
    // I_X: instance of X.
    $I_Y \Leftarrow \mathcal{R}(I_X)$
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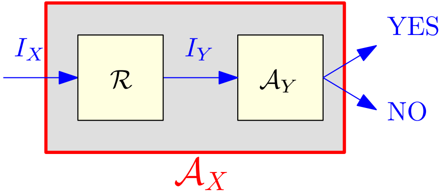


If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

Comparing Problems

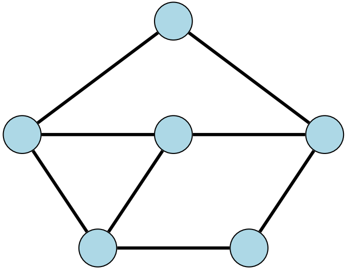
1. If there is reduction from X to Y...
2. "Problem X is no harder to solve than Problem Y".
3. If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
4. $X \leq Y$:
   1. X is no harder than Y, or
   2. Y is at least as hard as X.

Part II

Examples of Reductions

Independent Sets and Cliques

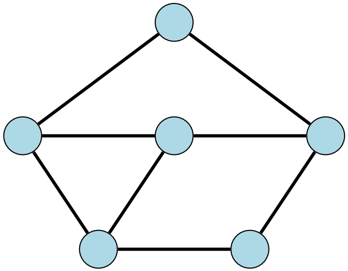
Given a graph G, a set of vertices V' is:

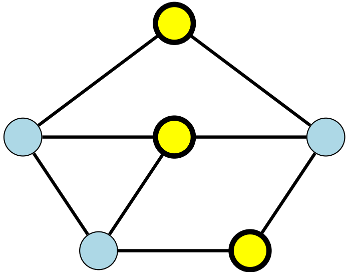


Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

1. ***independent set***: no two vertices of V' connected by an edge.

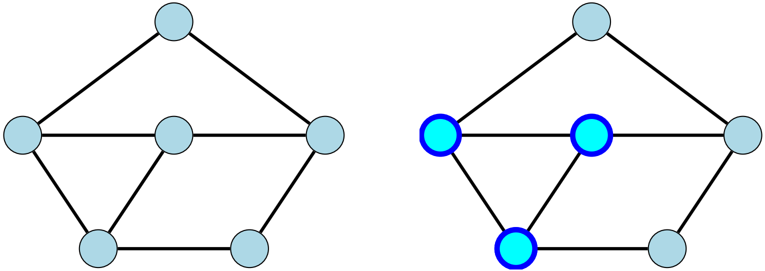




Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

1. ***independent set***: no two vertices of V' connected by an edge.
2. ***clique***: *every* pair of vertices in V' is connected by an edge of G.



The **Independent Set** and **Clique** Problems

Independent Set

Instance	G		k	
Question	G			$\geq k$

The **Independent Set** and **Clique** Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

Instance: A graph G and an integer k.
Question: Does G has a clique of size $\geq k$?

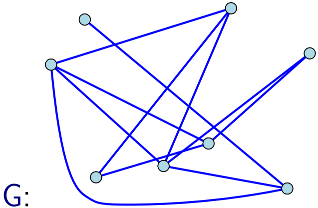
Recall

For decision problems X, Y, a reduction from X to Y is:

1. An algorithm ...
2. that takes I_X, an instance of X as input ...
3. and returns I_Y, an instance of Y as output ...
4. such that the solution (YES/NO) to I_Y is the same as the solution to I_X.

Reducing **Independent Set** to **Clique**

An instance of **Independent Set** is a graph G and an integer k.

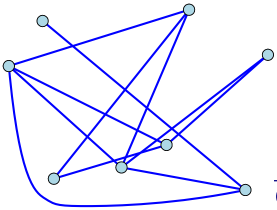


Reducing **Independent Set** to **Clique**

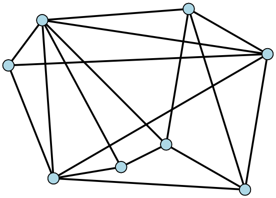
An instance of **Independent Set** is a graph $\boldsymbol{G}$ and an integer $\boldsymbol{k}$.

Reduction given $< \boldsymbol{G}, \boldsymbol{k} >$ outputs $< \overline{\boldsymbol{G}}, \boldsymbol{k} >$ where $\overline{\boldsymbol{G}}$ is the *complement* of $\boldsymbol{G}$. $\overline{\boldsymbol{G}}$ has an edge $(\boldsymbol{u}, \boldsymbol{v})$ if and only if $(\boldsymbol{u}, \boldsymbol{v})$ is not an edge of $\boldsymbol{G}$.

G:



$\overline{\text{G}}$:



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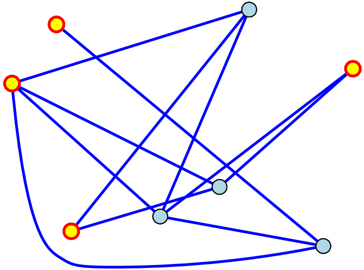

Independent set in G.

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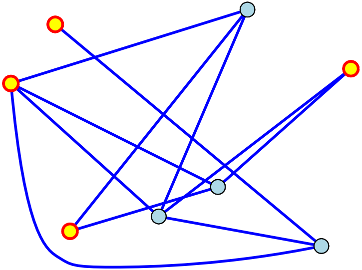

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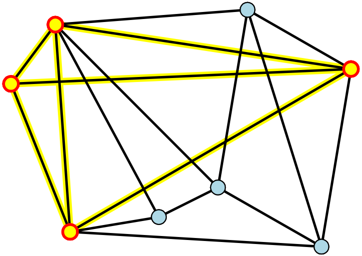

Clique in $\overline{G}$

Correctness of reduction

Lemma

G has an independent set of size k if and only if $\overline{G}$ has a clique of size k.

Proof.

Need to prove two facts:

G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.

$\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.

Easy to see both from the fact that $S \subseteq V$ is an independent set in G if and only if S is a clique in $\overline{G}$. □

Independent Set and Clique

1. **Independent Set** $\leq$ **Clique**.
   What does this mean?
2. If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
3. **Clique** is *at least as hard as* **Independent Set**.
4. Also... **Clique** $\leq$ **Independent Set**. Why? Thus **Clique** and **Independent Set** are polnomial-time equivalent.

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Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

DFA Universality

A DFA M is universal if it accepts every string.
That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA M.*
Goal: *Is M universal?*

How do we solve **DFA Universality**?
We check if M has *any* reachable non-final state.

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How do we solve **NFA Universality**?

Reduce it to **DFA Universality**?

Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.

The reduction takes exponential time!

NFA Universality is known to be PSPACE-Complete and we do not expect a polynomial-time algorithm.

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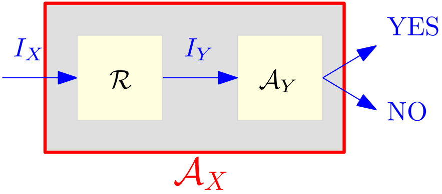
Polynomial-time reductions

We say that an algorithm is ***efficient*** if it runs in polynomial-time.

To find efficient algorithms for problems, we are only interested in polynomial-time reductions. Reductions that take longer are not useful.

If we have a polynomial-time reduction from problem X to problem Y (we write $X \leq_P Y$), and a poly-time algorithm $\mathcal{A}_Y$ for Y, we have a polynomial-time/efficient algorithm for X.



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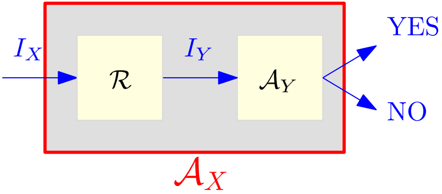

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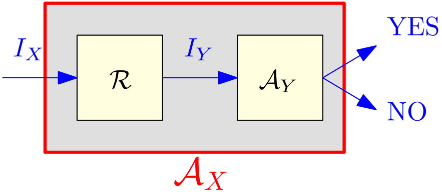

Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

1. given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$.
3. Answer to I_X YES *iff* answer to I_Y is YES.

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a ***Karp reduction***. Most reductions we will need are Karp reductions.Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.

Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

(A) Y can be solved in polynomial time.

(B) Y can NOT be solved in polynomial time.

(C) If Y is hard then X is also hard.

(D) None of the above.

(E) All of the above.

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

If you believe that **Independent Set** does not have an efficient algorithm, why should you believe the same of **Clique**?

Because we showed **Independent Set** $\leq_P$ **Clique**. If **Clique** had an efficient algorithm, so would **Independent Set**!

If $X \leq_P Y$ and X does not have an efficient algorithm, Y cannot have an efficient algorithm!

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Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{R}$ be a polynomial-time reduction from X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{R}$ is polynomial in the size of I_X.

Proof.

$\mathcal{R}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.

I_Y is the output of $\mathcal{R}$ on input I_X.

$\mathcal{R}$ can write at most $p(|I_X|)$ bits and hence $|I_Y| \leq p(|I_X|)$. □

Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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3. Answer to I_X YES *iff* answer to I_Y is YES.

Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Transitivity of Reductions

Proposition

$X \le_P Y$ and $Y \le_P Z$ implies that $X \le_P Z$.

Note: $X \le_P Y$ does not imply that $Y \le_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \le_P Y$ you need to show a reduction FROM X TO Y
That is, show that an algorithm for Y implies an algorithm for X.