Circuit satisfiability and Cook-Levin Theorem

Lecture 25

Thursday, December 7, 2017

LaTeXed: December 7, 2017 12:49

25.1: Recap

Recap

NP: languages that have non-deterministic polynomial time algorithms

A language L is **NP-Complete** iff

- L is in **NP**
- for every L' in **NP**, $L' \leq_P L$

L is **NP-Hard** if for every L' in **NP**, $L' \leq_P L$.

Theorem (Cook-Levin)

SAT is **NP-Complete**.

Recap

NP: languages that have non-deterministic polynomial time algorithms

A language L is **NP-Complete** iff

- L is in **NP**
- for every L' in **NP**, $L' \leq_P L$

L is **NP-Hard** if for every L' in **NP**, $L' \leq_P L$.

Theorem (Cook-Levin)

SAT is **NP-Complete**.

Recap

NP: languages that have non-deterministic polynomial time algorithms

A language L is **NP-Complete** iff

- L is in **NP**
- for every L' in **NP**, $L' \leq_P L$

L is **NP-Hard** if for every L' in **NP**, $L' \leq_P L$.

Theorem (Cook-Levin)

SAT is **NP-Complete***.*

Recap

NP: languages that have non-deterministic polynomial time algorithms

A language L is **NP-Complete** iff

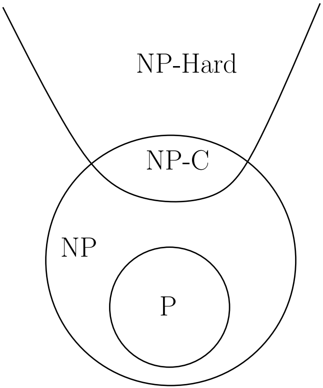
- L is in **NP**
- for every L' in **NP**, $L' \leq_P L$

L is **NP-Hard** if for every L' in **NP**, $L' \leq_P L$.

Theorem (Cook-Levin)

SAT is **NP-Complete**.

Pictorial View



P and NP

Possible scenarios:

1. $\mathbf{P} = \mathbf{NP}$.
2. $\mathbf{P} \neq \mathbf{NP}$

Question: Suppose $\mathbf{P} \neq \mathbf{NP}$. Is every problem in $\mathbf{NP} \setminus \mathbf{P}$ also **NP-Complete**?

Theorem (Ladner)

If $\mathbf{P} \neq \mathbf{NP}$ then there is a problem/language $X \in \mathbf{NP} \setminus \mathbf{P}$ such that X is not **NP-Complete**.

P and **NP**

Possible scenarios:

1. $\mathbf{P} = \mathbf{NP}$.
2. $\mathbf{P} \neq \mathbf{NP}$

Question: Suppose $\mathbf{P} \neq \mathbf{NP}$. Is every problem in $\mathbf{NP} \setminus \mathbf{P}$ also **NP-Complete**?

Theorem (Ladner)

If $\mathbf{P} \neq \mathbf{NP}$ then there is a problem/language $\boldsymbol{X} \in \mathbf{NP} \setminus \mathbf{P}$ such that $\boldsymbol{X}$ is not **NP-Complete**.

P and NP

Possible scenarios:

1. **P** $=$ **NP**.
2. **P** $\neq$ **NP**

Question: Suppose **P** $\neq$ **NP**. Is every problem in **NP** $\setminus$ **P** also **NP-Complete**?

Theorem (Ladner)

If **P** $\neq$ **NP** *then there is a problem/language* $X \in$ **NP** $\setminus$ **P** *such that* X *is not* **NP-Complete**.

Today

NP-Completeness of three problems:

- **3**-Color
- Circuit SAT

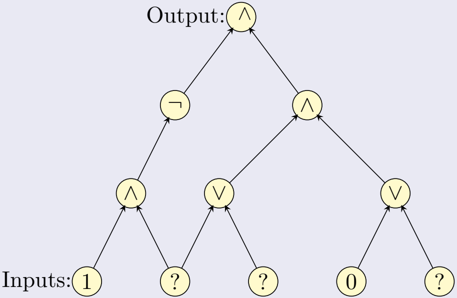
Important: understanding the problems and that they are hard.

Proofs and reductions will be sketchy and mainly to give a flavor

25.2: Circuit SAT

Definition

A circuit is a directed *acyclic* graph with



1. Input vertices (without incoming edges) labelled with **0**, **1** or a distinct variable.
2. Every other vertex is labelled $\vee$, $\wedge$ or $\neg$.
3. Single node output vertex with no outgoing edges.

CSAT: Circuit Satisfaction

Definition (Circuit Satisfaction (**CSAT**).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value **1**?

Claim

CSAT *is in* **NP**.

1. Certificate: Assignment to input variables.
2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

CSAT: Circuit Satisfaction

Definition (Circuit Satisfaction (CSAT).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value **1**?

Claim

CSAT *is in* **NP**.

1. Certificate: Assignment to input variables.
2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

Circuit SAT vs SAT

CNF formulas are a rather restricted form of Boolean formulas.

Circuits are a much more powerful (and hence easier) way to express Boolean formulas

However they are equivalent in terms of polynomial-time solvability.

Theorem

SAT $\leq_P$ *3SAT* $\leq_P$ *CSAT*.

Theorem

CSAT $\leq_P$ *SAT* $\leq_P$ *3SAT*.

Circuit SAT vs SAT

CNF formulas are a rather restricted form of Boolean formulas.

Circuits are a much more powerful (and hence easier) way to express Boolean formulas

However they are equivalent in terms of polynomial-time solvability.

Theorem

SAT $\leq_P$ ***3SAT*** $\leq_P$ ***CSAT***.

Theorem

CSAT $\leq_P$ ***SAT*** $\leq_P$ ***3SAT***.

Converting a CNF formula into a Circuit

3SAT $\leq_P$ CSAT

Given 3CNF formula φ with n variables and m clauses, create a Circuit C.

- Inputs to C are the n boolean variables $x_1, x_2, \ldots, x_n$
- Use NOT gate to generate literal $\neg x_i$ for each variable x_i
- For each clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ use two OR gates to mimic formula
- Combine the outputs for the clauses using AND gates to obtain the final output

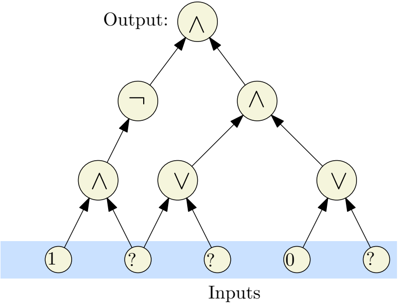
Example

$3SAT \leq_P CSAT$

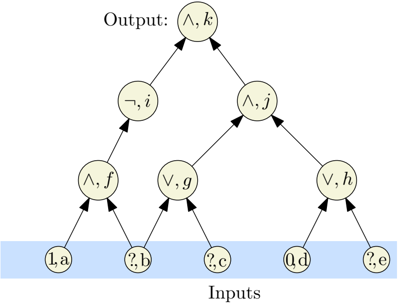
$$\varphi = \left(x_1 \vee \vee x_3 \vee x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4\right)$$

Converting a circuit into a CNF formula

Label the nodes



(A) Input circuit



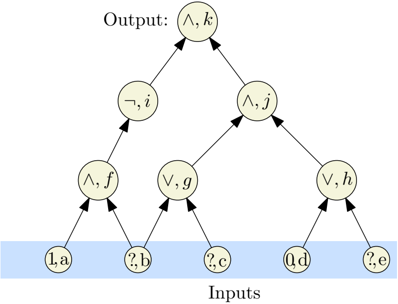
(B) Label the nodes.

The other direction: **CSAT $\leq_P$ 3SAT**

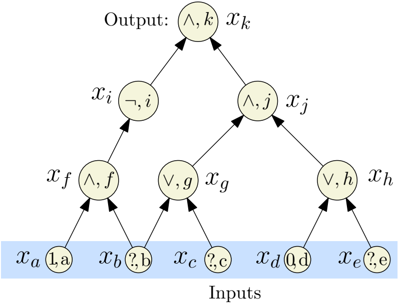
1. Now: **CSAT $\leq_P$ SAT**
2. More "interesting" direction.

Converting a circuit into a CNF formula

Introduce a variable for each node



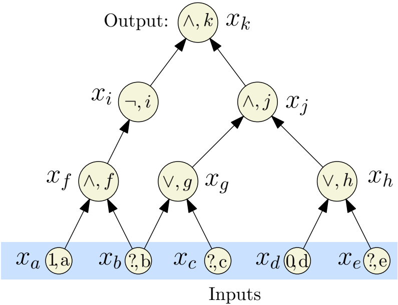
(B) Label the nodes.



(C) Introduce var for each node.

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.



(C) Introduce var for each node.

$\mathbf{x_k}$ (Demand a sat' assignment!)

$\mathbf{x_k = x_i \wedge x_j}$

$\mathbf{x_j = x_g \wedge x_h}$

$\mathbf{x_i = \neg x_f}$

$\mathbf{x_h = x_d \vee x_e}$

$\mathbf{x_g = x_b \vee x_c}$

$\mathbf{x_f = x_a \wedge x_b}$

$\mathbf{x_d = 0}$

$\mathbf{x_a = 1}$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

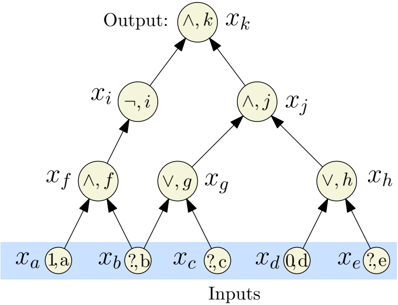
Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

x_k	x_k
$x_k = x_i \wedge x_j$	$(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$
$x_j = x_g \wedge x_h$	$(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h)$
$x_i = \neg x_f$	$(x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f)$
$x_h = x_d \vee x_e$	$(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)$
$x_g = x_b \vee x_c$	$(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$
$x_f = x_a \wedge x_b$	$(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)$
$x_d = 0$	$\neg x_d$
$x_a = 1$	x_a

Converting a circuit into a CNF formula

Take the conjunction of all the CNF sub-formulas



$$\begin{aligned}
&x_k \wedge (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \\
&\wedge (x_k \vee \neg x_i \vee \neg x_j) \wedge (\neg x_j \vee x_g) \\
&\wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \\
&\wedge (x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f) \\
&\wedge (x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \\
&\wedge (\neg x_h \vee x_d \vee x_e) \wedge (x_g \vee \neg x_b) \\
&\wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c) \\
&\wedge (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \\
&\wedge (x_f \vee \neg x_a \vee \neg x_b) \wedge (\neg x_d) \wedge x_a
\end{aligned}$$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.

Reduction: CSAT $\leq_P$ SAT

1. For each gate (vertex) v in the circuit, create a variable x_v
2. Case $\neg$: v is labeled $\neg$ and has one incoming edge from u (so $x_v = \neg x_u$). In **SAT** formula generate, add clauses $(x_u \vee x_v)$, $(\neg x_u \vee \neg x_v)$. Observe that

$$x_v = \neg x_u \text{ is true} \iff \begin{matrix} (x_u \vee x_v) \\ (\neg x_u \vee \neg x_v) \end{matrix} \text{ both true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

1. Case $\vee$: So $x_v = x_u \vee x_w$. In **SAT** formula generated, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$. Again, observe that

$$\left(x_v = x_u \vee x_w\right) \text{ is true} \quad \Longleftrightarrow \quad \begin{array}{l} (x_v \vee \neg x_u), \\ (x_v \vee \neg x_w), \\ (\neg x_v \vee x_u \vee x_w) \end{array} \quad \text{all true.}$$

1. Case $\wedge$: So $x_v = x_u \wedge x_w$. In SAT formula generated, add clauses $(\neg x_v \vee x_u)$, $(\neg x_v \vee x_w)$, and $(x_v \vee \neg x_u \vee \neg x_w)$. Again observe that

$$x_v = x_u \wedge x_w \text{ is true} \iff \begin{matrix} (\neg x_v \vee x_u), \\ (\neg x_v \vee x_w), \\ (x_v \vee \neg x_u \vee \neg x_w) \end{matrix} \quad \text{all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

1. If v is an input gate with a fixed value then we do the following. If $x_v = 1$ add clause x_v. If $x_v = 0$ add clause $\neg x_v$
2. Add the clause x_v where v is the variable for the output gate

Correctness of Reduction

Need to show circuit C is satisfiable iff φ_C is satisfiable

$\Rightarrow$ Consider a satisfying assignment a for C

1. Find values of all gates in C under a
2. Give value of gate v to variable x_v; call this assignment a'
3. a' satisfies φ_C (exercise)

$\Leftarrow$ Consider a satisfying assignment a for φ_C

1. Let a' be the restriction of a to only the input variables
2. Value of gate v under a' is the same as value of x_v in a
3. Thus, a' satisfies C

List of NP-Complete Problems to Remember

Problems

1. **SAT**
2. **3SAT**
3. **CircuitSAT**
4. **Independent Set**
5. **Clique**
6. **Vertex Cover**
7. **Hamilton Cycle** and **Hamilton Path** in both directed and undirected graphs
8. **3Color** and **Color**

25.3: NP-Completeness of Graph Coloring

Graph Coloring

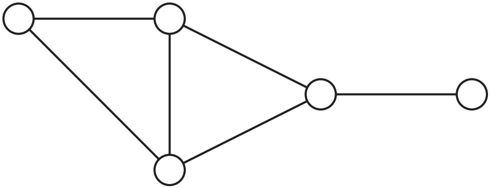
Problem: Graph Coloring

Instance: $G = (V, E)$: Undirected graph, integer k.
Question: Can the vertices of the graph be colored using k colors so that vertices connected by an edge do not get the same color?

Graph 3-Coloring

Problem: 3 Coloring

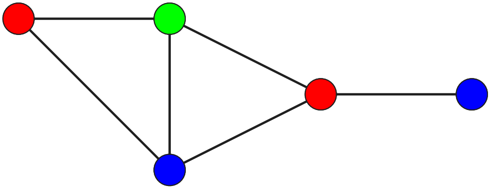
Instance: $G = (V, E)$: Undirected graph.
Question: Can the vertices of the graph be colored using 3 colors so that vertices connected by an edge do not get the same color?



Graph 3-Coloring

Problem: **3 Coloring**

Instance: $G = (V, E)$: Undirected graph.
Question: Can the vertices of the graph be colored using 3 colors so that vertices connected by an edge do not get the same color?



Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

Graph Coloring

1. Observation: If G is colored with k colors then each color class (nodes of same color) form an independent set in G.
2. G can be partitioned into k independent sets iff G is k-colorable.
3. Graph 2-Coloring can be decided in polynomial time.
4. G is 2-colorable iff G is bipartite!
5. There is a linear time algorithm to check if G is bipartite using **BFS** (we saw this earlier).

25.3.1: Problems related to graph coloring

Graph Coloring and Register Allocation

Register Allocation

Assign variables to (at most) k registers such that variables needed at the same time are not assigned to the same register

Interference Graph

Vertices are variables, and there is an edge between two vertices, if the two variables are "live" at the same time.

Observations

- [Chaitin] Register allocation problem is equivalent to coloring the interference graph with k colors
- Moreover, **3-COLOR** $\leq_P$ **k-Register Allocation**, for any $k \geq 3$

Class Room Scheduling

1. Given n classes and their meeting times, are k rooms sufficient?
2. Reduce to Graph k-Coloring problem
3. Create graph G
   - a node v_i for each class i
   - an edge between v_i and v_j if classes i and j *conflict*
4. Exercise: G is k-colorable iff k rooms are sufficient.

Class Room Scheduling

1. Given n classes and their meeting times, are k rooms sufficient?
2. Reduce to Graph k-Coloring problem
3. Create graph G
   - a node v_i for each class i
   - an edge between v_i and v_j if classes i and j *conflict*
4. Exercise: G is k-colorable iff k rooms are sufficient.

Class Room Scheduling

1. Given n classes and their meeting times, are k rooms sufficient?
2. Reduce to Graph k-Coloring problem
3. Create graph G
   - a node v_i for each class i
   - an edge between v_i and v_j if classes i and j *conflict*
4. Exercise: G is k-colorable iff k rooms are sufficient.

Class Room Scheduling

1. Given n classes and their meeting times, are k rooms sufficient?
2. Reduce to Graph k-Coloring problem
3. Create graph G
   - a node v_i for each class i
   - an edge between v_i and v_j if classes i and j *conflict*
4. Exercise: G is k-colorable iff k rooms are sufficient.

Class Room Scheduling

1. Given n classes and their meeting times, are k rooms sufficient?
2. Reduce to Graph k-Coloring problem
3. Create graph G
   - a node v_i for each class i
   - an edge between v_i and v_j if classes i and j *conflict*
4. Exercise: G is k-colorable iff k rooms are sufficient.

Frequency Assignments in Cellular Networks

1. Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)
   - Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
   - Each cell phone tower (simplifying) gets one band
   - Constraint: nearby towers cannot be assigned same band, otherwise signals will interference
2. Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?
3. Can reduce to k-coloring by creating interference/conflict graph on towers.

Frequency Assignments in Cellular Networks

1. Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)
   - Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
   - Each cell phone tower (simplifying) gets one band
   - Constraint: nearby towers cannot be assigned same band, otherwise signals will interference
2. Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?
3. Can reduce to k-coloring by creating interference/conflict graph on towers.

Frequency Assignments in Cellular Networks

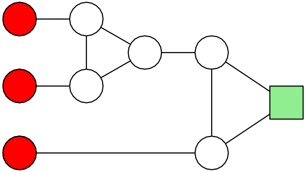
1. Cellular telephone systems that use Frequency Division Multiple Access (FDMA) (example: GSM in Europe and Asia and AT&T in USA)
    - Breakup a frequency range $[a, b]$ into disjoint *bands* of frequencies $[a_0, b_0], [a_1, b_1], \ldots, [a_k, b_k]$
    - Each cell phone tower (simplifying) gets one band
    - Constraint: nearby towers cannot be assigned same band, otherwise signals will interference
2. Problem: given k bands and some region with n towers, is there a way to assign the bands to avoid interference?
3. Can reduce to k-coloring by creating interference/conflict graph on towers.

25.4: Showing hardness of **3 COLORING**

3 color this gadget.

Clicker question

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming the two nodes are already colored as indicated).



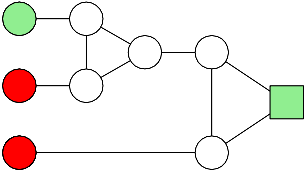
(A) Yes.

(B) No.

3 color this gadget II

Clicker question

You are given three colors: red, green and blue. Can the following graph be three colored in a valid way (assuming the two nodes are already colored as indicated).



(A) Yes.

(B) No.

3-Coloring is **NP-Complete**

- **3-Coloring** is in **NP**.
  - Certificate: for each node a color from $\{1, 2, 3\}$.
  - Certifier: Check if for each edge (u, v), the color of u is different from that of v.
- Hardness: We will show **3-SAT** $\leq_P$ **3-Coloring**.

Reduction Idea

1. φ: Given **3SAT** formula (i.e., 3CNF formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
   - create triangle with node True, False, Base
   - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
   - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., 3CNF formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
   - create triangle with node True, False, Base
   - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
   - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., $3\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
   - create triangle with node True, False, Base
   - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
   - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., $\mathbf{3}\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
    - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
    - create triangle with node True, False, Base
    - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
    - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
    - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., $3\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
    - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
    - create triangle with node True, False, Base
    - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
    - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
    - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., $3\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
   - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
   - create triangle with node True, False, Base
   - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
   - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
   - Need to add constraints to ensure clauses are satisfied (next phase)

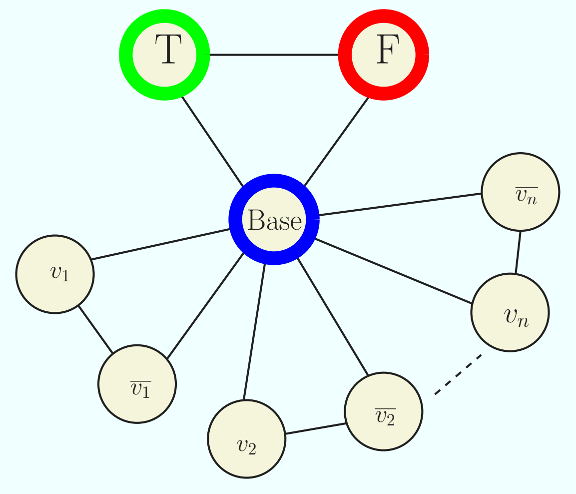
Reduction Idea

1. φ: Given **3SAT** formula (i.e., $3\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
    - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
    - create triangle with node True, False, Base
    - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
    - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
    - Need to add constraints to ensure clauses are satisfied (next phase)

Reduction Idea

1. φ: Given **3SAT** formula (i.e., $3\mathrm{CNF}$ formula).
2. φ: variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.
3. Create graph G_φ s.t. G_φ 3-colorable $\iff$ φ satisfiable.
    - encode assignment $x_1, \ldots, x_n$ in colors assigned nodes of G_φ.
    - create triangle with node True, False, Base
    - for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
    - If graph is 3-colored, either v_i or $\bar{v}_i$ gets the same color as True. Interpret this as a truth assignment to v_i
    - Need to add constraints to ensure clauses are satisfied (next phase)

Figure

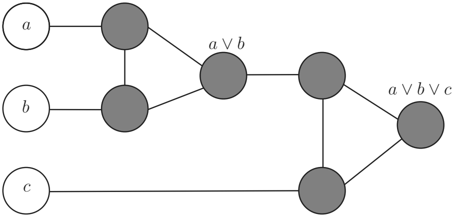


Clause Satisfiability Gadget

1. For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph
   - gadget graph connects to nodes corresponding to a, b, c
   - needs to implement OR
2. OR-gadget-graph:

Clause Satisfiability Gadget

1. For each clause $C_j = (a \vee b \vee c)$, create a small gadget graph
   - gadget graph connects to nodes corresponding to a, b, c
   - needs to implement OR
2. OR-gadget-graph:



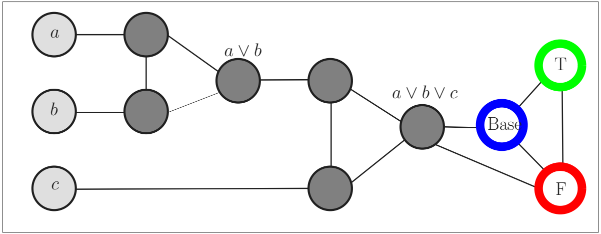
OR-Gadget Graph

Property: if a, b, c are colored False in a 3-coloring then output node of OR-gadget has to be colored False.

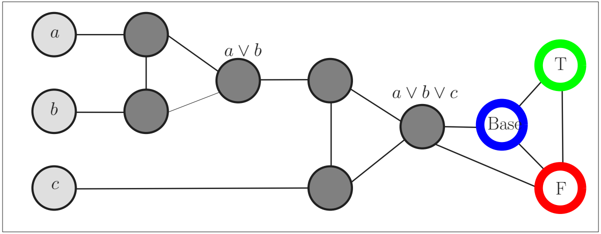
Property: if one of a, b, c is colored True then OR-gadget can be 3-colored such that output node of OR-gadget is colored True.

Reduction

- create triangle with nodes True, False, Base
- for each variable x_i two nodes v_i and $\bar{v}_i$ connected in a triangle with common Base
- for each clause $C_j = (a \vee b \vee c)$, add OR-gadget graph with input nodes a, b, c and connect output node of gadget to both False and Base



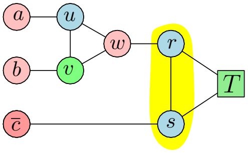
Reduction



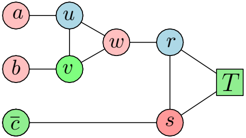
Claim

No legal **3***-coloring of above graph (with coloring of nodes* ***T*****,** ***F*****,** ***B*** *fixed) in which* ***a*****,** ***b*****,** ***c*** *are colored False. If any of* ***a*****,** ***b*****,** ***c*** *are colored True then there is a legal* **3***-coloring of above graph.*

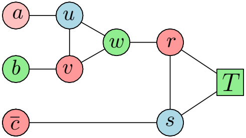
3 coloring of the clause gadget



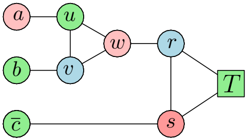
FFF - **BAD**



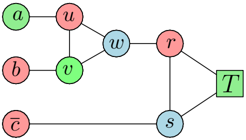
FFT



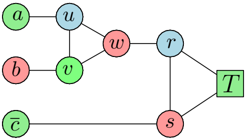
FTF



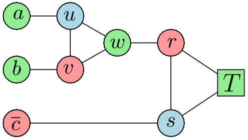
FTT



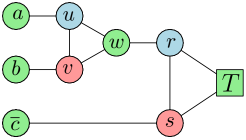
TFF



TFT



TTF

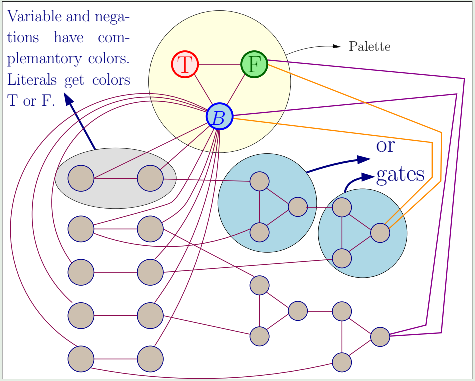


TTT

Reduction Outline

Example

$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$



Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \lor b \lor c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \lor b \lor c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Correctness of Reduction

φ is satisfiable implies G_φ is 3-colorable

- if x_i is assigned True, color v_i True and $\bar{v}_i$ False
- for each clause $C_j = (a \vee b \vee c)$ at least one of a, b, c is colored True. OR-gadget for C_j can be 3-colored such that output is True.

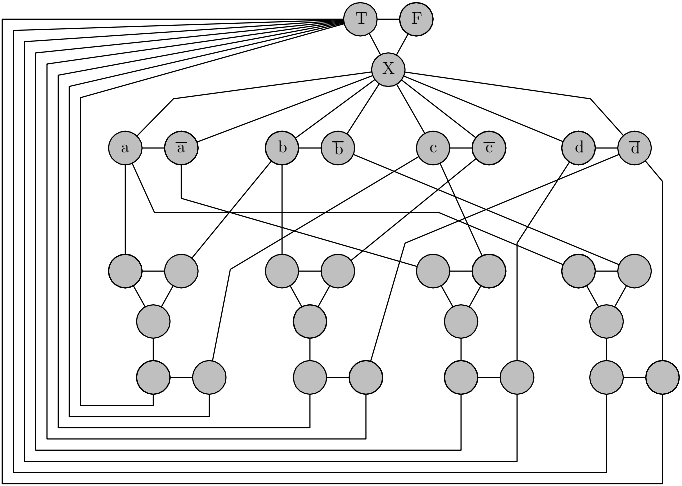
G_φ is 3-colorable implies φ is satisfiable

- if v_i is colored True then set x_i to be True, this is a legal truth assignment
- consider any clause $C_j = (a \vee b \vee c)$. it cannot be that all a, b, c are False. If so, output of OR-gadget for C_j has to be colored False but output is connected to Base and False!

Graph generated in reduction...

... from 3SAT to 3COLOR

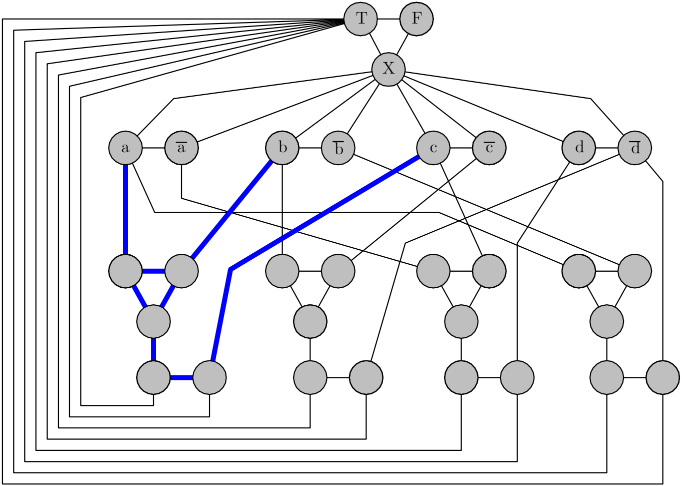
$$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$$



Graph generated in reduction...

... from 3SAT to 3COLOR

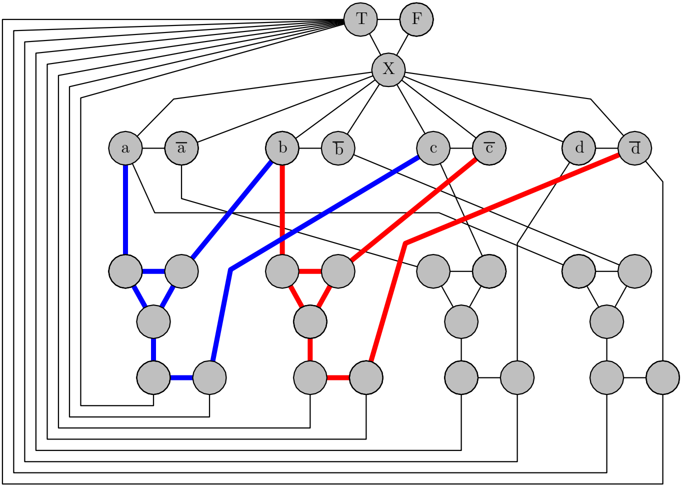
$$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$$



Graph generated in reduction...

... from 3SAT to 3COLOR

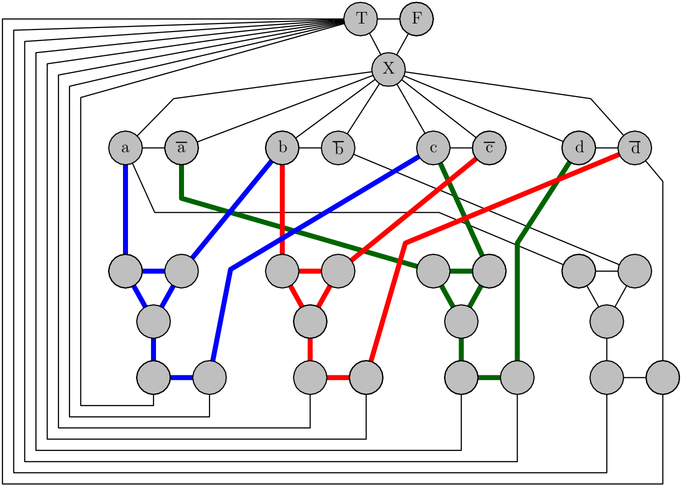
$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$



Graph generated in reduction...

... from 3SAT to 3COLOR

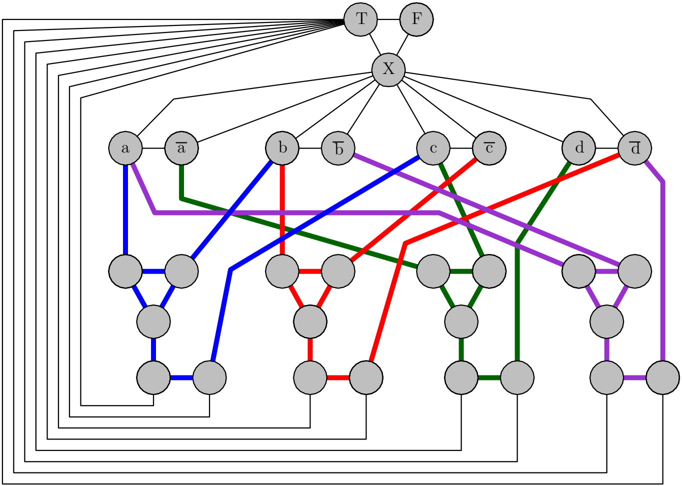
$$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$$



Graph generated in reduction...

... from 3SAT to 3COLOR

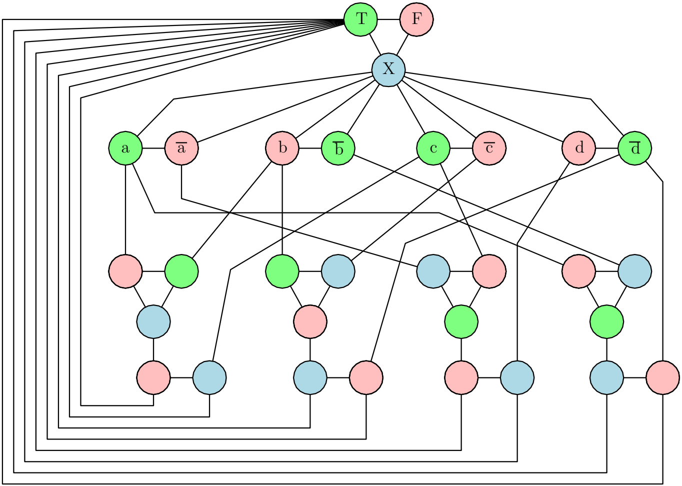
$$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$$



Graph generated in reduction...

... from 3SAT to 3COLOR

$$(a \vee b \vee c) \wedge \left(b \vee \overline{c} \vee \overline{d}\right) \wedge (\overline{a} \vee c \vee d) \wedge \left(a \vee \overline{b} \vee \overline{d}\right)$$



25.5: Proof of Cook-Levin Theorem

Cook-Levin Theorem

Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

We have already seen that **SAT** is in **NP**.

Need to prove that *every* language $L \in$ **NP**, $L \leq_P$ **SAT**

Difficulty: Infinite number of languages in **NP**. Must *simultaneously* show a *generic* reduction strategy.

Cook-Levin Theorem

Theorem (Cook-Levin)

SAT *is* **NP-Complete***.*

We have already seen that **SAT** is in **NP**.

Need to prove that *every* language $L \in \mathbf{NP}$, $L \leq_P \mathbf{SAT}$

Difficulty: Infinite number of languages in **NP**. Must *simultaneously* show a *generic* reduction strategy.

High-level Plan

What does it mean that $L \in \mathbf{NP}$?

$L \in NP$ implies that there is a non-deterministic TM M and polynomial $p()$ such that

$$L = \{x \in \Sigma^* \mid M \text{ accepts } x \text{ in at most } p(|x|) \text{ steps}\}$$

We will describe a reduction f_M that depends on M, p such that:

- f_M takes as input a string x and outputs a SAT formula $f_M(x)$
- f_M runs in time polynomial in $|x|$
- $x \in L$ if and only if $f_M(x)$ is satisfiable

High-level Plan

What does it mean that $L \in$ **NP**?

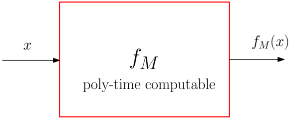
$L \in NP$ implies that there is a non-deterministic TM M and polynomial $p()$ such that

$$L = \{x \in \Sigma^* \mid M \text{ accepts } x \text{ in at most } p(|x|) \text{ steps}\}$$

We will describe a reduction f_M that depends on M, p such that:

- f_M takes as input a string x and outputs a SAT formula $f_M(x)$
- f_M runs in time polynomial in $|x|$
- $x \in L$ if and only if $f_M(x)$ is satisfiable

Plan continued



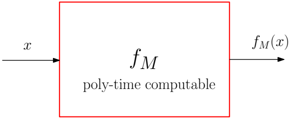
$f_M(x)$ is satisfiable if and only if $x \in L$
$f_M(x)$ is satisfiable if and only if nondeterministic M accepts x in $p(|x|)$ steps

BIG IDEA

- $f_M(x)$ will express "M on input x accepts in $p(|x|)$ steps"
- $f_M(x)$ will encode a computation history of M on x

$f_M(x)$ will be a carefully constructed CNF formula s.t if we have a satisfying assignment to it, then we will be able to see a complete accepting computation of M on x *down to the last detail* of where the head is, what transition is chosen, what the tape contents are, at each step.

Plan continued



$f_M(x)$ is satisfiable if and only if $x \in L$
$f_M(x)$ is satisfiable if and only if nondeterministic M accepts x in $p(|x|)$ steps

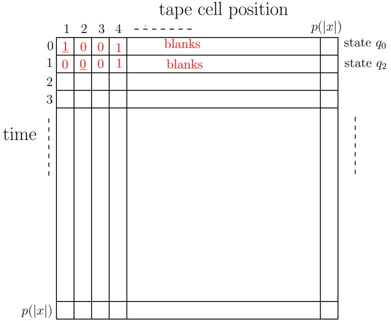
BIG IDEA

- $f_M(x)$ will express "M on input x accepts in $p(|x|)$ steps"
- $f_M(x)$ will encode a computation history of M on x

$f_M(x)$ will be a carefully constructed CNF formula s.t if we have a satisfying assignment to it, then we will be able to see a complete accepting computation of M on x *down to the last detail* of where the head is, what transition is chosen, what the tape contents are, at each step.

Tableau of Computation

M runs in time $p(|x|)$ on x. Entire computation of M on x can be represented by a “tableau”



Row i gives contents of all cells at time i

At time 0 tape has input x followed by blanks

Each row long enough to hold all cells M might ever have scanned.

Variable of $f_M(x)$

Four types of variable to describe computation of M on x

- $T(b, h, i)$: tape cell at position h holds symbol b at time i.
  $1 \leq h \leq p(|x|)$, $b \in \Gamma$, $0 \leq i \leq p(|x|)$
- $H(h, i)$: read/write head is at position h at time i.
  $1 \leq h \leq p(|x|)$, $0 \leq i \leq p(|x|)$
- $S(q, i)$ state of M is q at time i $q \in Q$, $0 \leq i \leq p(|x|)$
- $I(j, i)$ instruction number j is executed at time i
  M is non-deterministic, need to specify transitions in some way.
  Number transitions as $1, 2, \ldots, \ell$ where jth transition is
  $< q_j, b_j, q'_j, b'_j, d_j >$ indication $(q'_j, b'_j, d_j) \in \delta(q_j, b_j)$,
  direction $d_j \in \{-1, 0, 1\}$.

Number of variables is $O(p(|x|)^2)$ where constant in $O()$ hides dependence on fixed machine M.

Notation

Some abbreviations for ease of notation

$\bigwedge_{k=1}^{m} x_k$ means $x_1 \wedge x_2 \wedge \ldots \wedge x_m$

$\bigvee_{k=1}^{m} x_k$ means $x_1 \vee x_2 \vee \ldots \vee x_m$

$\bigoplus(x_1, x_2, \ldots, x_k)$ is a formula that means exactly one of $x_1, x_2, \ldots, x_m$ is true. Can be converted to CNF form

Clauses of $f_M(x)$

$f_M(x)$ is the conjunction of 8 clause groups:

$$f_M(x) = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_6 \wedge \varphi_7 \wedge \varphi_8$$

where each φ_i is a CNF formula. Described in subsequent slides.
Property: $f_M(x)$ is satisfied iff there is a truth assignment to the variables that simultaneously satisfy $\varphi_1, \ldots, \varphi_8$.

φ_1

φ_1 asserts (is true iff) the variables are set T/F indicating that M starts in state q_0 at time 0 with tape contents containing x followed by blanks.

Let $x = a_1 a_2 \ldots a_n$

$\varphi_1 = S(q, 0)$ state at time 0 is q_0

$\bigwedge$ and

$\bigwedge_{h=1}^{n} T(a_h, h, 0)$ at time 0 cells 1 to n have a_1 to a_n

$\bigwedge_{h=n+1}^{p(|x|}) T(B, h, 0)$ at time 0 cells $n+1$ to $p(|x|)$ have blanks

$\bigwedge$ and

$H(1, 0)$ head at time 0 is in position 1

φ_2

φ_2 asserts M in exactly one state at any time i

$$\varphi_2 = \bigwedge_{i=0}^{p(|x|)} \left(\oplus(S(q_0, i), S(q_1, i), \ldots, S(q_{|Q|}, i)) \right)$$

φ_3

φ_3 asserts that each tape cell holds a unique symbol at any given time.

$$\varphi_3 = \bigwedge_{i=0}^{p(|x|)} \bigwedge_{h=1}^{p(|x|)} \oplus\left(T(b_1, h, i), T(b_2, h, i), \ldots, T(b_{|\Gamma|}, h, i)\right)$$

For each time i and for each cell position h exactly one symbol $b \in \Gamma$ at cell position h at time i

φ_4

φ_4 asserts that the read/write head of M is in exactly one position at any time i

$$\varphi_4 = \bigwedge_{i=0}^{p(|x|)} \left(\oplus \left(H(1,i), H(2,i), \ldots, H(p(|x|), i) \right) \right)$$

φ_5

φ_5 asserts that M accepts

- Let q_a be unique accept state of M
- without loss of generality assume M runs all $p(|x|)$ steps

$$\varphi_5 = S(q_a, p(|x|))$$

State at time $p(|x|)$ is q_a the accept state.

If we don't want to make assumption of running for all steps

$$\varphi_5 = \bigvee_{i=1}^{p(|x|)} S(q_a, i)$$

which means M enters accepts state at some time.

φ_6

φ_6 asserts that M executes a unique instruction at each time

$$\varphi_6 = \bigwedge_{i=0}^{p(|x|)} \oplus(I(1,i), I(2,i), \ldots, I(m,i))$$

where m is max instruction number.

φ_7

φ_7 ensures that variables don't allow tape to change from one moment to next if the read/write head was not there.

"If head is **not** at position h at time i then at time $i + 1$ the symbol at cell h must be unchanged"

$$\varphi_7 = \bigwedge_i \bigwedge_h \bigwedge_{b \neq c} \left(\overline{H(h,i)} \Rightarrow \overline{T(b,h,i) \bigwedge T(c,h,i+1)} \right)$$

since $A \Rightarrow B$ is same as $\neg A \vee B$, rewrite above in CNF form

$$\varphi_7 = \bigwedge_i \bigwedge_h \bigwedge_{b \neq c} \left(H(h,i) \vee \neg T(b,h,i) \vee \neg T(c,h,i+1) \right)$$

φ_8

φ_8 asserts that changes in tableau/tape correspond to transitions of M (as Lenny says, this is the big cookie).

Let jth instruction be $< q_j, b_j, q'_j, b'_j, d_j >$

$\varphi_8 = \bigwedge_i \bigwedge_j (I(j,i) \Rightarrow S(q_j, i))$ If instr j executed at time i then state must be correct to do j $\bigwedge$

$\bigwedge_i \bigwedge_j (I(j,i) \Rightarrow S(q'_j, i+1))$ and at next time unit, state must be the proper next state for instr j $\bigwedge$

$\bigwedge_i \bigwedge_h \bigwedge_j [(I(j,i) \bigwedge H(h,i)) \Rightarrow T(b_j, h, i)]$ if j was executed and head was at position h, then cell h has correct symbol for j $\bigwedge$

$\bigwedge_i \bigwedge_j \bigwedge_h [(I(j,i) \wedge H(h,i)) \Rightarrow T(b'_j, h, i+1)]$ if j was done then at time i with head at h then at next time step symbol b'_j was indeed written in position h $\bigwedge$

$\bigwedge_i \bigwedge_j \bigwedge_h [(I(j,i) \wedge H(h,i)) \Rightarrow H(h + d_j, i+1)]$ and head is moved properly according to instr j.

Proof of Correctness

(Sketch)

- Given M, x, poly-time algorithm to construct $f_M(x)$
- if $f_M(x)$ is satisfiable then the truth assignment completely specifies an accepting computation of M on x
- if M accepts x then the accepting computation leads to an "obvious" truth assignment to $f_M(x)$. Simply assign the variables according to the state of M and cells at each time i.

Thus M accepts x if and only if $f_M(x)$ is satisfiable