Algorithms & Models of Computation

CS/ECE 374, Fall 2017

Non-deterministic Finite Automata (NFAs)

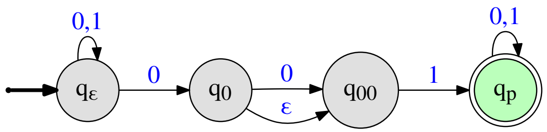
Lecture 4

Thursday, September 7, 2017

Part I

NFA Introduction

Non-deterministic Finite State Automata (NFAs)



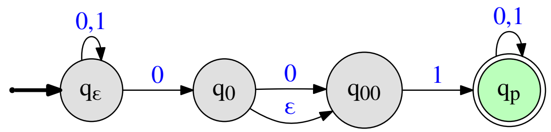
Differences from DFA

- From state q on same letter $a \in \Sigma$ multiple possible states
- No transitions from q on some letters
- ε-transitions!

Questions:

- Is this a "real" machine?
- What does it do?

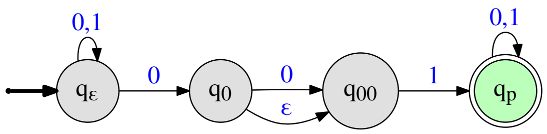
NFA behavior



Machine on input string w from state q can lead to set of states (could be empty)

- From q_ε on $\mathbf{1}$
- From q_ε on $\mathbf{0}$
- From q_0 on ε
- From q_ε on $\mathbf{01}$
- From q_{00} on $\mathbf{00}$

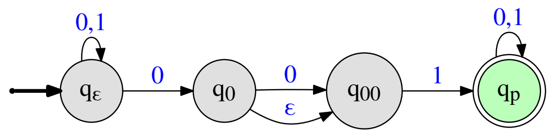
NFA acceptance: informal



Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

The language accepted (or recognized) by a NFA N is denote by $L(N)$ and defined as: $L(N) = \{w \mid N \text{ accepts } w\}$.

NFA acceptance: example



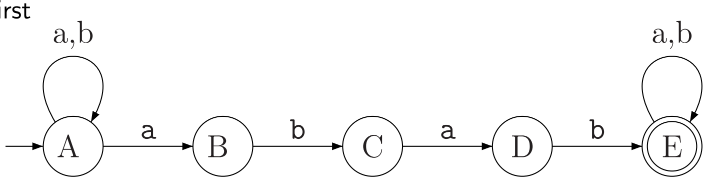
- Is **01** accepted?
- Is **001** accepted?
- Is **100** accepted?
- Are all strings in $\mathbf{1^*01}$ accepted?
- What is the language accepted by N?

Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is **not** accepted.

Simulating NFA

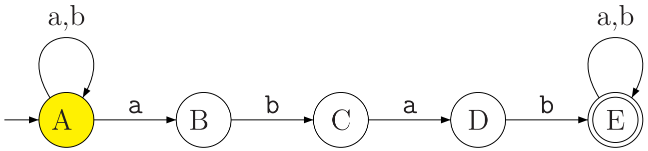
Example the first

(N1)



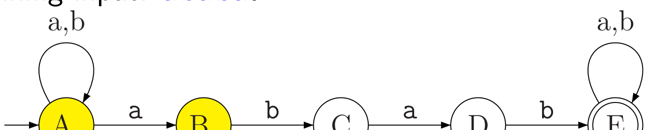
Run it on input ***ababa***.
Idea: Keep track of the states where the NFA might be at any given time.

$t = 0$:



Remaining input: ***ababa***.

$t = 1$:



Remaining input: ***baba***.

Formal Tuple Notation

Definition

A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- Q is a finite set whose elements are called states,
- Σ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \cup \{\varepsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function (here $\mathcal{P}(Q)$ is the power set of Q),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\varepsilon\}$ is a subset of Q — a set of states.

Reminder: Power set

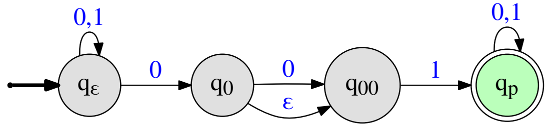
For a set Q its power set is: $\mathcal{P}(Q) = 2^Q = \{X \mid X \subseteq Q\}$ is the set of all subsets of Q.

Example

$Q = \{1, 2, 3, 4\}$

$$\mathcal{P}(Q) = \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3\}, \\ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1\}, \{2\}, \{3\}, \{4\}, \\ \{\} \end{array} \right\}$$

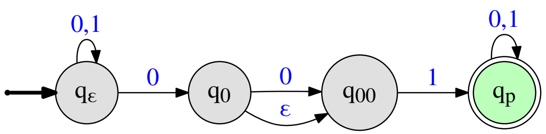
Example



- $Q = \{q_\varepsilon, q_0, q_{00}, q_p\}$
- $\Sigma = \{0, 1\}$
- δ
- $s = q_\varepsilon$
- $A = \{q_p\}$

Example

Transition function in detail...



$\delta(q_\varepsilon, \varepsilon) = \{q_\varepsilon\}$
$\delta(q_\varepsilon, 0) = \{q_\varepsilon, q_0\}$
$\delta(q_\varepsilon, 1) = \{q_\varepsilon\}$

$\delta(q_0, \varepsilon) = \{q_0, q_{00}\}$
$\delta(q_0, 0) = \{q_{00}\}$
$\delta(q_0, 1) = \{\}$

$\delta(q_{00}, \varepsilon) = \{q_{00}\}$
$\delta(q_{00}, 0) = \{\}$
$\delta(q_{00}, 1) = \{q_p\}$

$\delta(q_p, \varepsilon) = \{q_p\}$
$\delta(q_p, 0) = \{q_p\}$
$\delta(q_p, 1) = \{q_p\}$

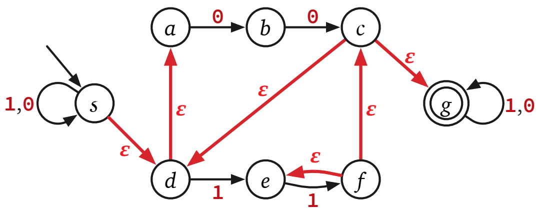
Extending the transition function to strings

1. NFA $N = (Q, \Sigma, \delta, s, A)$
2. $\delta(q, a)$: set of states that N can go to from q on reading $a \in \Sigma \cup \{\varepsilon\}$.
3. Want transition function $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$
4. $\delta^*(q, w)$: set of states reachable on input w starting in state q.

Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the ϵreach(q) is the set of all states that q can reach using only ε-transitions.



Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the ϵreach(q) is the set of all states that q can reach using only ε-transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $\delta^*(q, a) = \cup_{p \in \epsilon\text{reach}(q)} \left(\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r)\right)$
- if $w = ax$,
  $\delta^*(q, w) = \cup_{p \in \epsilon\text{reach}(q)} \left(\cup_{r \in \delta(p,a)} \delta^*(r, x)\right)$

Formal definition of language accepted by N

Definition

A string w is accepted by NFA N if $\delta^*_N(s, w) \cap A \neq \emptyset$.

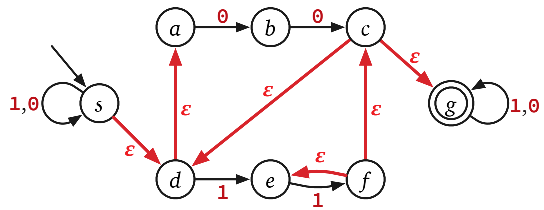
Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Important: Formal definition of the language of NFA above uses δ^* and not δ. As such, one does not need to include ε-transitions closure when specifying δ, since δ^* takes care of that.

Example



What is:

- $\delta^*(s, \epsilon)$
- $\delta^*(s, 0)$
- $\delta^*(c, 0)$
- $\delta^*(b, 00)$

Another definition of computation

Definition

$q \xrightarrow{w}_N p$: State p of NFA N is **reachable** from q on w $\iff$ there exists a sequence of states $r_0, r_1, \ldots, r_k$ and a sequence $x_1, x_2, \ldots, x_k$ where $x_i \in \Sigma \cup \{\varepsilon\}$, for each i, such that:

- $r_0 = q$,
- for each i, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = p$, and
- $w = x_1 x_2 x_3 \cdots x_k$.

Definition

$$\delta^* N(q, w) = \left\{ p \in Q \;\middle|\; q \xrightarrow{w}_N p \right\}.$$

Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to "design" programs
- Fundamental in **theory** to prove many theorems
- Very important in **practice** directly and indirectly
- Many deep connections to various fields in Computer Science and Mathematics

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.

Part II

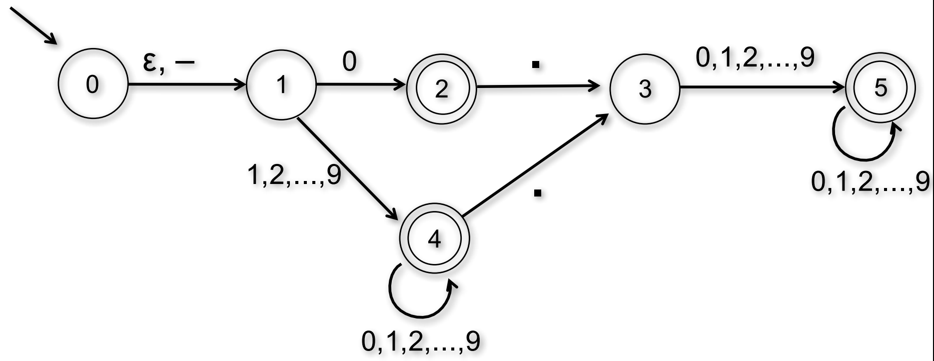
Constructing NFAs

DFAs and NFAs

- Every DFA is a NFA so NFAs are at least as powerful as DFAs.
- NFAs prove ability to "guess and verify" which simplifies design and reduces number of states
- Easy proofs of some closure properties

Example

Strings that represent decimal numbers.



Example

- {strings that contain CS374 as a substring}
- {strings that contain CS374 or CS473 as a substring}
- {strings that contain CS374 and CS473 as substrings}

Example

$L_k = \{$bitstrings that have a 1 k positions from the end$\}$

A simple transformation

Theorem

For every NFA N *there is another* NFA N' *such that* $L(N) = L(N')$ *and such that* N' *has the following two properties:*

- N' *has single final state* f *that has no outgoing transitions*
- *The start state* s *of* N *is different from* f

Part III

Closure Properties of NFAs

Closure properties of NFAs

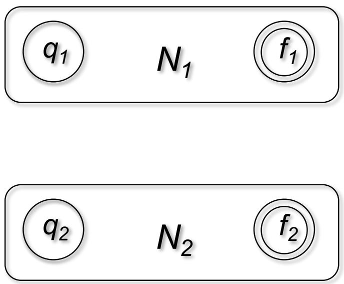
Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement

Closure under union

Theorem

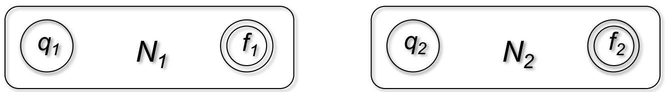
For any two NFAs N_1 and N_2 there is a NFA N such that $L(N) = L(N_1) \cup L(N_2)$.



Closure under concatenation

Theorem

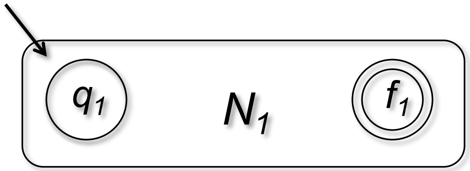
For any two NFAs N_1 and N_2 there is a NFA N such that $L(N) = L(N_1) \bullet L(N_2)$.



Closure under Kleene star

Theorem

For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



Closure under Kleene star

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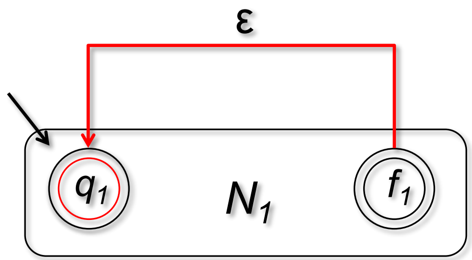

Does not work! Why?

Closure under Kleene star

Theorem

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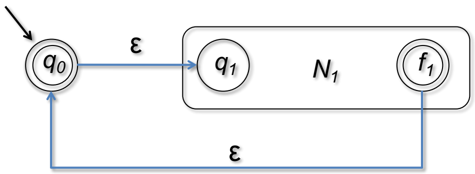


Part IV

NFAs capture Regular Languages

Regular Languages Recap

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	**a** denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1 r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

NFAs and Regular Language

Theorem

For every regular language L there is an NFA N such that $L = L(N)$.

Proof strategy:

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

NFAs and Regular Language

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

Base cases: $\emptyset$, $\{\varepsilon\}$, $\{a\}$ for $a \in \Sigma$.

NFAs and Regular Language

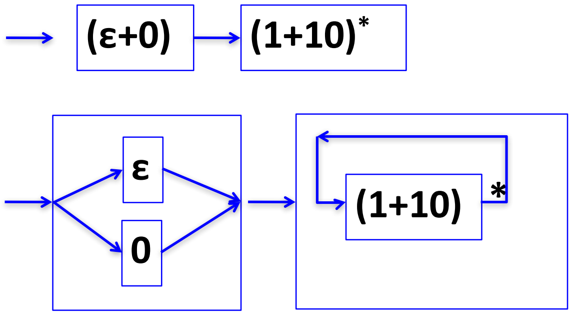
- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

Inductive cases:

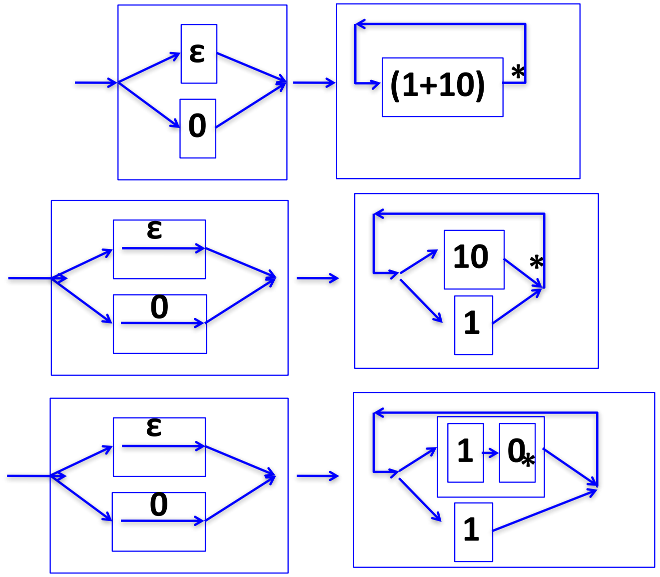
- r_1, r_2 regular expressions and $r = r_1 + r_2$.
  By induction there are NFAs N_1, N_2 s.t $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$. We have already seen that there is NFA N s.t $L(N) = L(N_1) \cup L(N_2)$, hence $L(N) = L(r)$
- $r = r_1 \bullet r_2$. Use closure of NFA languages under concatenation
- $r = (r_1)^*$. Use closure of NFA languages under Kleene star

Example

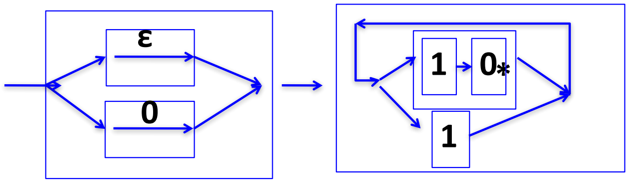
$(\varepsilon+0)(1+10)^*$

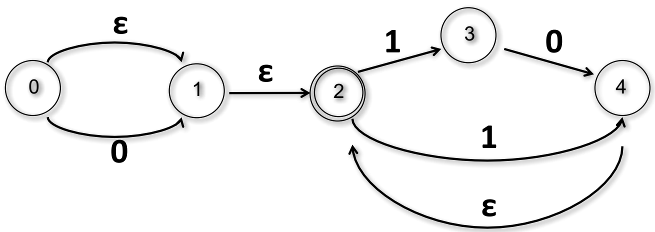


Example



Example





Final NFA simplified slightly to reduce states