Prove that each of the following languages is ***not*** regular.

1. $\left\{\mathtt{0}^{2^n} \;\middle|\; n \geq 0\right\}$

Solution (verbose): Let $F = L = \left\{\mathtt{0}^{2^n} \;\middle|\; n \geq 0\right\}$.

Let x and y be arbitrary elements of F.

Then $x = \mathtt{0}^{2^i}$ and $y = \mathtt{0}^{2^j}$ for some non-negative integers x and y.

Let $z = \mathtt{0}^{2^i}$.

Then $xz = \mathtt{0}^{2^i}\mathtt{0}^{2^i} = \mathtt{0}^{2^{i+1}} \in L$.

And $yz = \mathtt{0}^{2^j}\mathtt{0}^{2^i} = \mathtt{0}^{2^i+2^j} \notin L$, because $i \neq j$

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings $\mathtt{0}^{2^i}$ and $\mathtt{0}^{2^j}$ are distinguished by the suffix $\mathtt{0}^{2^i}$, because $\mathtt{0}^{2^i}\mathtt{0}^{2^i} = \mathtt{0}^{2^{i+1}} \in L$ but $\mathtt{0}^{2^j}\mathtt{0}^{2^i} = \mathtt{0}^{2^{i+j}} \notin L$. Thus L itself is an infinite fooling set for L. ■

2. $\{\mathtt{0}^{2n}\mathtt{1}^n \mid n \geq 0\}$

Solution (verbose): Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{0}^i\mathtt{1}^i$.

Then $xz = \mathtt{0}^{2i}\mathtt{1}^i \in L$.

And $yz = \mathtt{0}^{i+j}\mathtt{1}^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^i$ and $\mathtt{0}^j$ are distinguished by the suffix $\mathtt{0}^i\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{i+j}\mathtt{1}^i \notin L$. Thus, the language $\mathtt{0}^*$ is an infinite fooling set for L. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^{2i}$ and $\mathtt{0}^{2j}$ are distinguished by the suffix $\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{2j}\mathtt{1}^i \notin L$. Thus, the language $(\mathtt{00})^*$ is an infinite fooling set for L. ■

3. $\{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$

Solution (verbose): Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{0}^i\mathtt{1}^i$.

Then $xz = \mathtt{0}^{2i}\mathtt{1}^i \notin L$.

And $yz = \mathtt{0}^{i+j}\mathtt{1}^i \in L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^{2i}$ and $\mathtt{0}^{2j}$ are distinguished by the suffix $\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \notin L$ but $\mathtt{0}^{2j}\mathtt{1}^i \in L$. Thus, the language $(\mathtt{00})^*$ is an infinite fooling set for L. ■

4. Strings over $\{\mathtt{0},\mathtt{1}\}$ where the number of $\mathtt{0}$s is exactly twice the number of $\mathtt{1}$s.

Solution (verbose): Let F be the language $\mathtt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \mathtt{0}^i\mathtt{1}^i$.

Then $xz = \mathtt{0}^{2i}\mathtt{1}^i \in L$.

And $yz = \mathtt{0}^{i+j}\mathtt{1}^i \notin L$, because $i + j \neq 2i$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise, different fooling set): For all non-negative integers $i \neq j$, the strings $\mathtt{0}^{2i}$ and $\mathtt{0}^{2j}$ are distinguished by the suffix $\mathtt{1}^i$, because $\mathtt{0}^{2i}\mathtt{1}^i \in L$ but $\mathtt{0}^{2j}\mathtt{1}^i \notin L$. Thus, the language $(\mathtt{00})^*$ is an infinite fooling set for L. ■

Solution (closure properties): If L were regular, then the language

$$\big((\mathtt{0}+\mathtt{1})^* \setminus L\big) \cap \mathtt{0}^*\mathtt{1}^* \;=\; \{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{\mathtt{0}^m\mathtt{1}^n \mid m \neq 2n\}$ is not regular in problem 3. *[Yes, this proof would be worth full credit, either in homework or on an exam.]* ■

5. Strings of properly nested parentheses `()`, brackets `[]`, and braces `{}`. For example, the string `([]){}` is in this language, but the string `([)]` is not, because the left and right delimiters don't match.

Solution (verbose): Let F be the language $\texttt{(}^*$.

Let x and y be arbitrary strings in F.

Then $x = \texttt{(}^i$ and $y = \texttt{(}^j$ for some non-negative integers $i \neq j$.

Let $z = \texttt{)}^i$.

Then $xz = \texttt{(}^i\texttt{)}^i \in L$.

And $yz = \texttt{(}^j\texttt{)}^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings $\texttt{(}^i$ and $\texttt{(}^j$ are distinguished by the suffix $\texttt{)}^i$, because $\texttt{(}^i\texttt{)}^i \in L$ but $\texttt{(}^i\texttt{)}^j \notin L$. Thus, the language $\texttt{(}^*$ is an infinite fooling set. ■

6. Strings of the form $w_1\texttt{\#}w_2\texttt{\#}\cdots\texttt{\#}w_n$ for some $n \geq 2$, where each substring w_i is a string in $\{\texttt{0},\texttt{1}\}^*$, and some pair of substrings w_i and w_j are equal.

Solution (verbose): Let F be the language $\texttt{0}^*$.

Let x and y be arbitrary strings in F.

Then $x = \texttt{0}^i$ and $y = \texttt{0}^j$ for some non-negative integers $i \neq j$.

Let $z = \texttt{\#0}^i$.

Then $xz = \texttt{0}^i\texttt{\#0}^i \in L$.

And $yz = \texttt{0}^j\texttt{\#0}^i \notin L$, because $i \neq j$.

Thus, F is a fooling set for L.

Because F is infinite, L cannot be regular. ■

Solution (concise): For any non-negative integers $i \neq j$, the strings $\texttt{0}^i$ and $\texttt{0}^j$ are distinguished by the suffix $\texttt{\#0}^i$, because $\texttt{0}^i\texttt{\#0}^i \in L$ but $\texttt{0}^j\texttt{\#0}^i \notin L$. Thus, the language $\texttt{0}^*$ is an infinite fooling set. ■

Work on these later:

7. $\left\{\mathtt{0}^{n^2} \;\middle|\; n \geq 0\right\}$

Solution: Let x and y be distinct arbitrary strings in L.

Without loss of generality, $x = \mathtt{0}^{i^2}$ and $y = \mathtt{0}^{j^2}$ for some $i > j \geq 0$.

Let $z = \mathtt{0}^{2j+1}$.

Then $yz = \mathtt{0}^{j^2+2j+1} = \mathtt{0}^{(j+1)^2} \in L$

On the other hand, $xz = \mathtt{0}^{i^2+2j+1} \notin L$, because $i^2 < i^2 + 2j + 1 < i^2 + 2i + 1 = (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that L is an infinite fooling set for L, so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in $\mathtt{0}^*$.

Without loss of generality, $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some $i > j \geq 0$.

Let $z = \mathtt{0}^{i^2+i+1}$.

Then $xz = \mathtt{0}^{i^2+2i+1} = \mathtt{0}^{(i+1)^2} \in L$.

On the other hand, $yz = \mathtt{0}^{i^2+i+j+1} \notin L$, because $i^2 < i^2+i+j+1 < i^2+2i+1 = (i+1)^2$.

Thus, z distinguishes x and y.

We conclude that $\mathtt{0}^*$ is an infinite fooling set for L, so L cannot be regular. ■

Solution: Let x and y be distinct arbitrary strings in $\mathtt{00}^*$.

Without loss of generality, $x = \mathtt{0}^i$ and $y = \mathtt{0}^j$ for some $i > j \geq 1$.

Let $z = \mathtt{0}^{i^2-i}$.

Then $xz = \mathtt{0}^{i^2} \in L$.

On the other hand, $yz = \mathtt{0}^{i^2-i+j} \notin L$, because

$$(i-1)^2 \;=\; i^2 - 2i + 1 \;<\; i^2 - i \;<\; i^2 - i + j \;<\; i^2.$$

(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.)

Thus, z distinguishes x and y.

We conclude that $\mathtt{00}^*$ is an infinite fooling set for L, so L cannot be regular. ■

8. $\{w \in (\mathtt{0}+\mathtt{1})^* \mid w \text{ is the binary representation of a perfect square}\}$

Solution: We design our fooling set around numbers of the form $(2^k+1)^2 = 2^{2k}+2^{k+1}+1 = \mathtt{10}^{k-2}\mathtt{10}^k\mathtt{1} \in L$, for any integer $k \geq 2$. The argument is somewhat simpler if we further restrict k to be even.

Let $F = \mathtt{1}(\mathtt{00})^*\mathtt{1}$, and let x and y be arbitrary strings in F.

Then $x = \mathtt{10}^{2i-2}\mathtt{1}$ and $y = \mathtt{10}^{2j-2}\mathtt{1}$, for some positive integers $i \neq j$.

Without loss of generality, assume $i < j$. (Otherwise, swap x and y.)

Let $z = \mathtt{0}^{2i}\mathtt{1}$.

Then $xz = \mathtt{10}^{2i-2}\mathtt{10}^{2i}\mathtt{1}$ is the binary representation of $2^{4i} + 2^{2i+1} + 1 = (2^{2i}+1)^2$, and therefore $xz \in L$.

On the other hand, $yz = \mathtt{10}^{2j-2}\mathtt{10}^{2i}\mathtt{1}$ is the binary representation of $2^{2i+2j} + 2^{2i+1} + 1$. Simple algebra gives us the inequalities

$$\begin{aligned}
(2^{i+j})^2 &= 2^{2i+2j} \\
&< \boldsymbol{2^{2i+2j} + 2^{2i+1} + 1} \\
&< 2^{2(i+j)} + 2^{i+j+1} + 1 \\
&= (2^{i+j}+1)^2.
\end{aligned}$$

So $2^{2i+2j} + 2^{2i+1} + 1$ lies between two consecutive perfect squares, and thus is not a perfect square, which implies that $yz \notin L$.

We conclude that F is a fooling set for L. Because F is infinite, L cannot be regular. ■