Regular Languages and Expressions

Lecture 2

August 30, 2018

Part I

Regular Languages

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A class of simple but very useful languages.

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Regular languages are closed under the operations of union, concatenation and Kleene star.

Some simple regular languages

Lemma

If w is a string then $L = \{w\}$ is regular.

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Lemma

Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

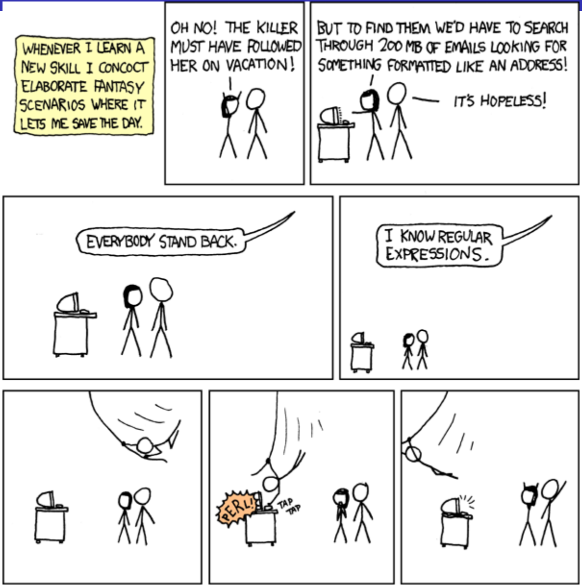
More Examples

- $\{w \mid w$ is a keyword in Python program$\}$
- $\{w \mid w$ is a valid date of the form mm/dd/yy$\}$
- $\{w \mid w$ describes a valid Roman numeral$\}$
  $\{I, II, III, IV, V, VI, VII, VIII, IX, X, XI, \ldots\}$.
- $\{w \mid w$ contains "CS374" as a substring$\}$.

Part II

Regular Expressions

xkcd



https://xkcd.com/208/

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star named after him.

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabhe $\mathbf{\Sigma}$ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
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Inductive cases: If $\mathbf{r_1}$ and $\mathbf{r_2}$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r_1} + \mathbf{r_2})$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r_1 r_2})$ denotes the language $R_1 R_2$
- $(\mathbf{r_1})^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	**Regular Expressions**
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	$\mathbf{a}$ denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r}_1 + \mathbf{r}_2$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r}_1\mathbf{r}_2$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression $\mathbf{r}$, $L(\mathbf{r})$ is the language denoted by $\mathbf{r}$. Multiple regular expressions can denote the same language!
  Example: $(0 + 1)$ and $(1 + 0)$ denote same language $\{0, 1\}$

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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

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- Given a regular expression $\mathbf{r}$ we would like to "understand" $L(\mathbf{r})$ (say by giving an English description)

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- $(\epsilon + \mathbf{0})(\mathbf{1} + \mathbf{10})^*$: strings without two consecutive 0s.

Creating regular expressions

- bitstrings with the pattern **001** or the pattern **100** ocurring as a substring

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  one answer: $\mathbf{(0+1)^*001(0+1)^* + (0+1)^*100(0+1)^*}$

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- bitstrings that do <u>not</u> contain **011** as a substring

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  $(1^*0^+ + 0^*)(10^+)^*(1+\epsilon)$

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- bitstrings that do not contain $\mathbf{011}$ as a substring
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  $(1^*0^+ + 0^*)(10^+)^*(1 + \epsilon)$
- Hard: bitstrings with an odd number of 1s and an odd number of 0s.

Regular expression identities

- $r^*r^* = r^*$ meaning for any regular expression r, $L(r^*r^*) = L(r^*)$
- $(r^*)^* = r^*$
- $rr^* = r^*r$
- $(rs)^*r = r(sr)^*$
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Question: How does on prove an identity?
By induction. On what? Length of r since r is a string obtained from specific inductive rules.

A non-regular language and other closure properties

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Theorem

L is **not** *a regular language.*

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Other questions:

- Suppose R_1 is regular and R_2 is regular. Is $R_1 \cap R_2$ regular?
- Suppose R_1 is regular is $\bar{R}_1$ (complement of R_1) regular?