Regular Languages and Expressions

Lecture 2

August 30, 2018

Part I

Regular Languages

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A class of simple but very useful languages.

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- If L is regular, then $L^* = \cup_{n \geq 0} L^n$ is regular

$L = \{0,1\}$ $L^* =$ $L = \{00, 01\}$

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Regular languages are closed under the operations of union, concatenation and Kleene star.

Some simple regular languages

Lemma

If w is a string then $L = \{w\}$ is regular.

Example: $\{aba\}$ or $\{abbabbab\}$. Why?

$\{a\} \cdot \{b\} \{a\} = \{aba\}$

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Lemma

Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

More Examples

- $\{w \mid w$ is a keyword in Python program$\}$
- $\{w \mid w$ is a valid date of the form mm/dd/yy$\}$
- $\{w \mid w$ describes a valid Roman numeral$\}$
  $\{I, II, III, IV, V, VI, VII, VIII, IX, X, XI, \ldots\}$.
- $\{w \mid w$ contains "CS374" as a substring$\}$.

Σ^*

$\Sigma = \{0, 1\}$

$\Sigma^* \cdot \{CS374\} \cdot \Sigma^*$

Part II

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him.

Inductive Definition

A regular expression **r** over an alphabhe $\mathbf{\Sigma}$ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- **a** denote the language $\{a\}$.

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Inductive cases: If $\mathbf{r_1}$ and $\mathbf{r_2}$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r_1} + \mathbf{r_2})$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r_1 r_2})$ denotes the language $R_1 R_2$
- $(\mathbf{r_1})^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages

$\emptyset$ regular
$\{\epsilon\}$ regular
$\{a\}$ regular for $a \in \Sigma$
$R_1 \cup R_2$ regular if both are
$R_1 R_2$ regular if both are
R^* is regular if R is

Regular Expressions

$\emptyset$ denotes $\emptyset$
ϵ denotes $\{\epsilon\}$
$\mathbf{a}$ denote $\{a\}$
$\mathbf{r}_1 + \mathbf{r}_2$ denotes $R_1 \cup R_2$
$\mathbf{r}_1\mathbf{r}_2$ denotes $R_1 R_2$
$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression $\mathbf{r}$, $L(\mathbf{r})$ is the language denoted by $\mathbf{r}$. Multiple regular expressions can denote the same language!
  Example: $(0 + 1)$ and $(1 + 0)$ denote same language $\{0, 1\}$

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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.

$L^* = \bigcup_{n \geq 0} L^n$

$L^+ = \bigcup_{n \geq 1} L^n$

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- Superscript $+$. For convenience, define $\mathbf{r}^+ = \mathbf{r}\mathbf{r}^*$. Hence if $L(\mathbf{r}) = R$ then $L(\mathbf{r}^+) = R^+$.
- Other notation: $r + s$, $r \cup s$, $r|s$ all denote union. rs is sometimes written as $r \bullet s$.

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- Given a language L "in mind" (say an English description) we would like to write a regular expression for L (if possible)
- Given a regular expression **r** we would like to "understand" $L(\mathbf{r})$ (say by giving an English description)

Understanding regular expressions

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- $(\epsilon + 1)(01)^*(\epsilon + 0)$:

$\{\varepsilon, 01, 0101, 010101,$

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- $(\epsilon+\mathbf{0})(\mathbf{1}+\mathbf{10})^*$:

{1, 10}

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- $\emptyset 0$: $\{\}$
- $(\epsilon + 1)(01)^*(\epsilon + 0)$: alteranting 0s and 1s. Alternatively, no two consecutive 0s and no two conescutive 1s
- $(\epsilon + 0)(1 + 10)^*$: strings without two consecutive 0s.

Creating regular expressions

- bitstrings with the parttern **001** or the patter **100** ocurring as a substring

$$(0+1)^* 001 (0+1)^*$$
$$+ \quad (0+1)^* 100 (0+1)^*$$
$$= (0+1)^* (001+100) (0+1)^*$$

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  one answer: $(0+1)^*001(0+1)^* + (0+1)^*100(0+1)^*$

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- bitstrings with an even number of $\mathbf{1}$'s

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  one answer: $\mathbf{0}^*\mathbf{1}r$ where r is solution to previous part

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- bitstrings that do *not* contain $\mathbf{011}$ as a substring

111   (01) 1^*0^*

0^*+1^* 10   0100100001000

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- bitstrings that do *not* contain **011** as a substring
  one answer: $0^*(10^+)^*(1+\epsilon)$

~~$(0^* + 1^*0)(1+0^+)^*$~~

11111 0

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- bitstrings that do *not* contain $\mathbf{011}$ as a substring
  one answer: $\mathbf{0^*(10^+)^*(1+\epsilon)}$
- Hard: bitstrings with an odd number of 1s *and* an odd number of 0s.

Regular expression identities

- $r^* r^* = r^*$ meaning for any regular expression r, $L(r^* r^*) = L(r^*)$
- $(r^*)^* = r^*$
- $rr^* = r^* r$
- $(rs)^* r = r(sr)^*$
- $(r + s)^* = (r^* s^*)^* = (r^* + s^*)^* = (r + s^*)^* = \dots$

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Question: How does on prove an identity?

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Question: How does on prove an identity?
By induction. On what? Length of r since r is a string obtained from specific inductive rules.

A non-regular language and other closure properties

Consider $L = \{0^n 1^n \mid n \geq 0\} = \{\epsilon, 01, 0011, 000111, \ldots\}$.

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Theorem

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How do we prove it?

Other questions:

- Suppose R_1 is regular and R_2 is regular. Is $R_1 \cap R_2$ regular?
- Suppose R_1 is regular is $\bar{R}_1$ (complement of R_1) regular?