NFAs continued, Closure Properties of Regular Languages

Lecture 5

September 11, 2018

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

- DFAs are special cases of NFAs (trivial)
- NFAs accept regular expressions (today)
- DFAs accept languages accepted by NFAs (today)
- Regular expressions for languages accepted by DFAs (later in the course)

Part I

Closure Properties of NFAs

A simple transformation

Theorem

For every NFA N there is another NFA N' such that $L(N) = L(N')$ and such that N' has the following two properties:

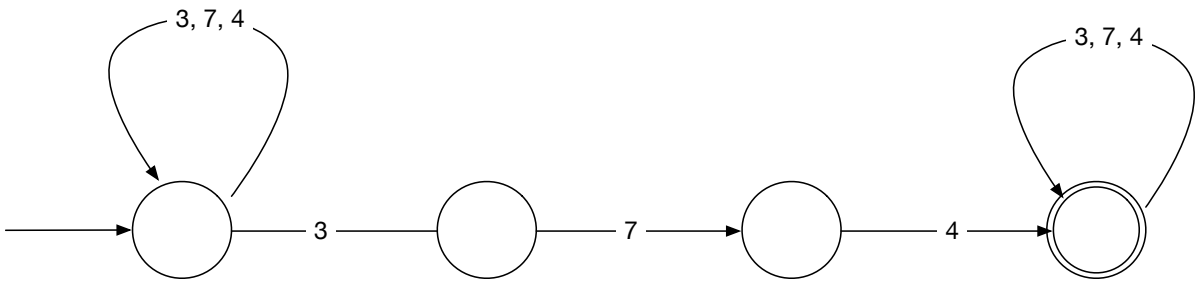
- *N' has single final state f that has no outgoing transitions*
- *The start state s of N is different from f*

Closure Properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

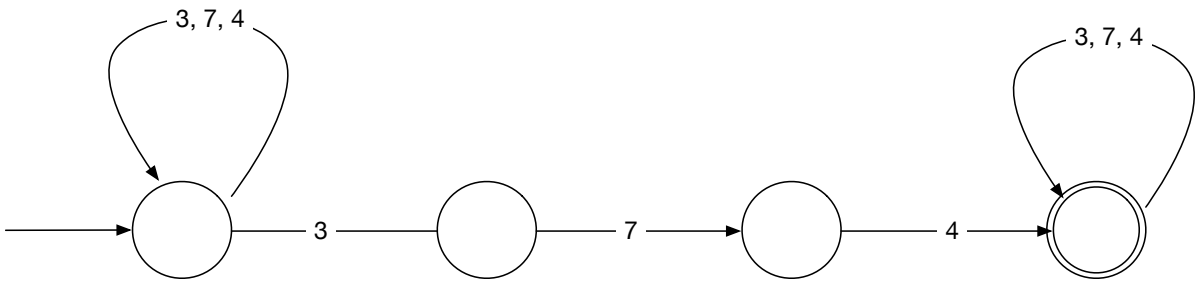
- union
- intersection
- concatenation
- Kleene star
- complement

Example

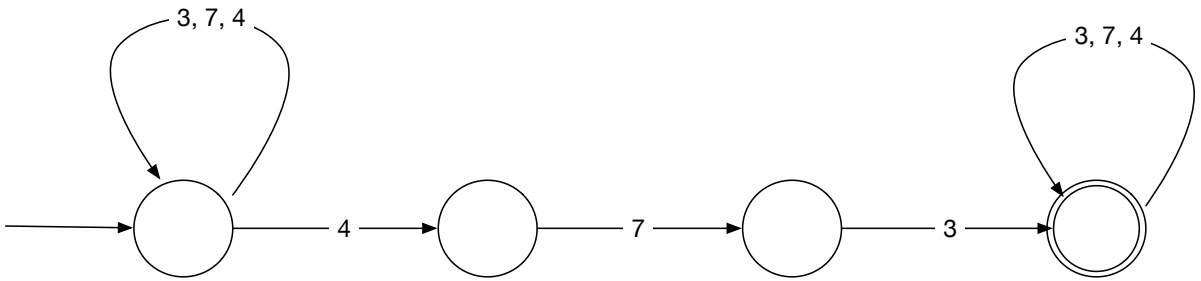


All strings that contain the substring **374**

Example

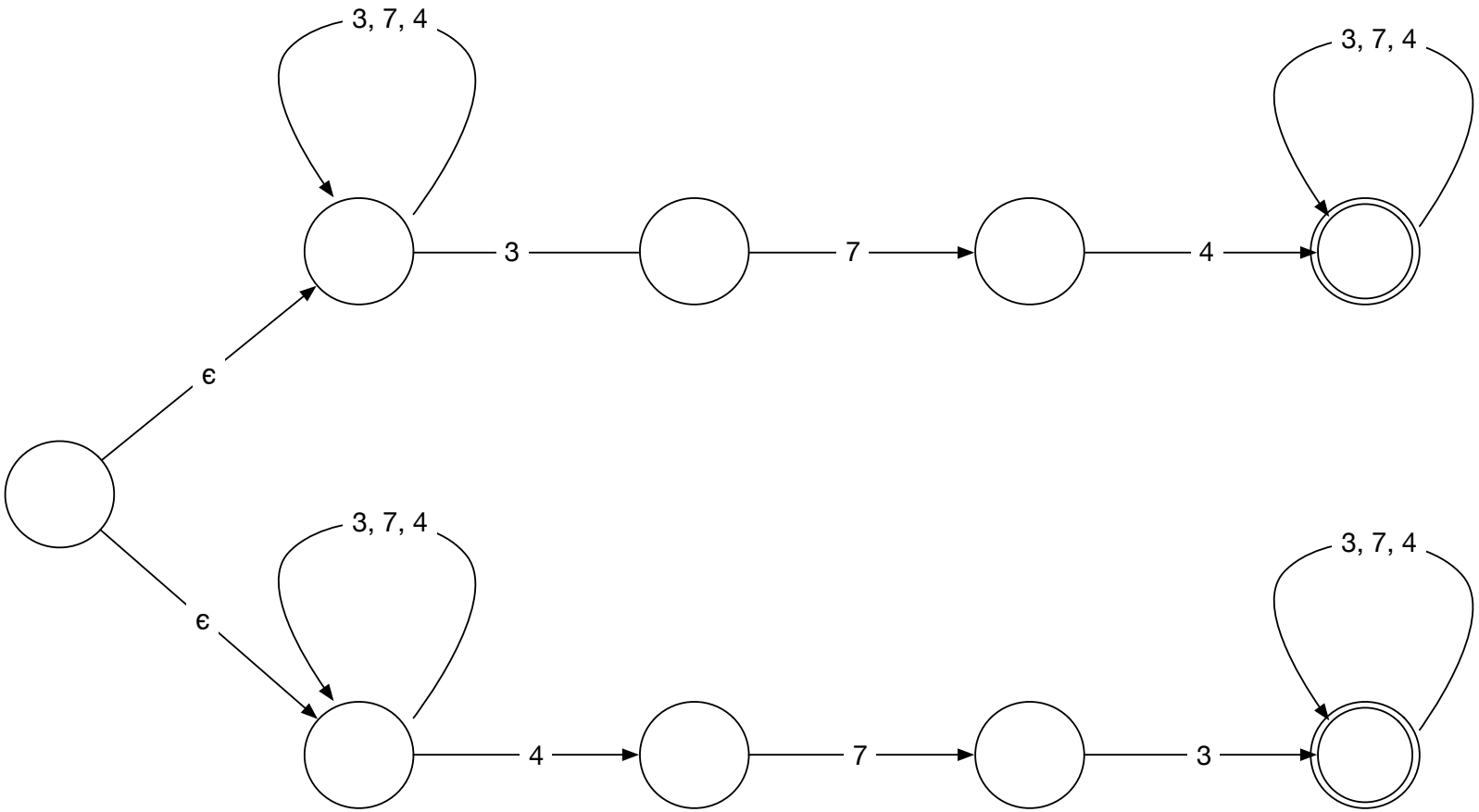


All strings that contain the substring **374**



All strings that contain the substring **473**

Example II



All strings that contain either **374** or **473**

Closure under union

Theorem

For any two NFAs N_1 *and* N_2 *there is a NFA* N *such that* $L(N) = L(N_1) \cup L(N_2)$.

Closure under union

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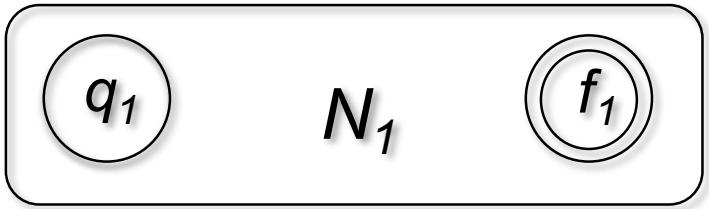

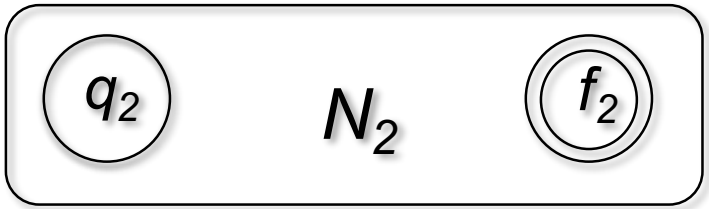

Closure under concatenation

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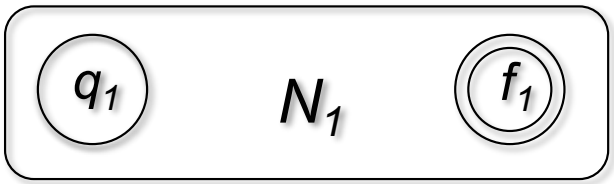
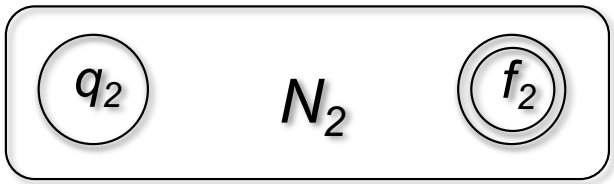
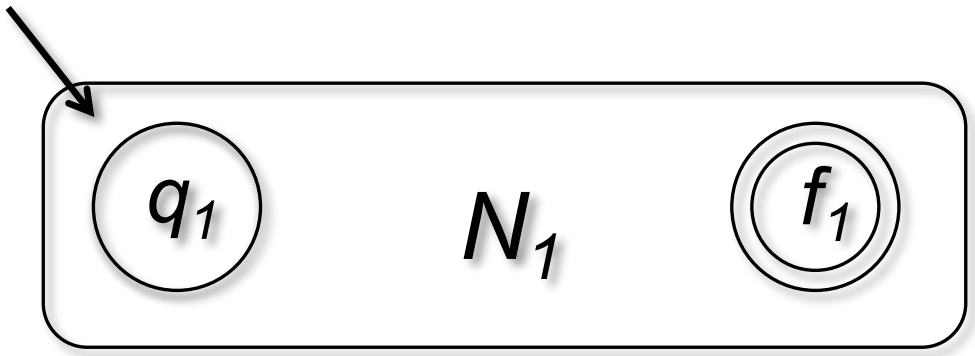




Closure under Kleene star

Theorem

For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



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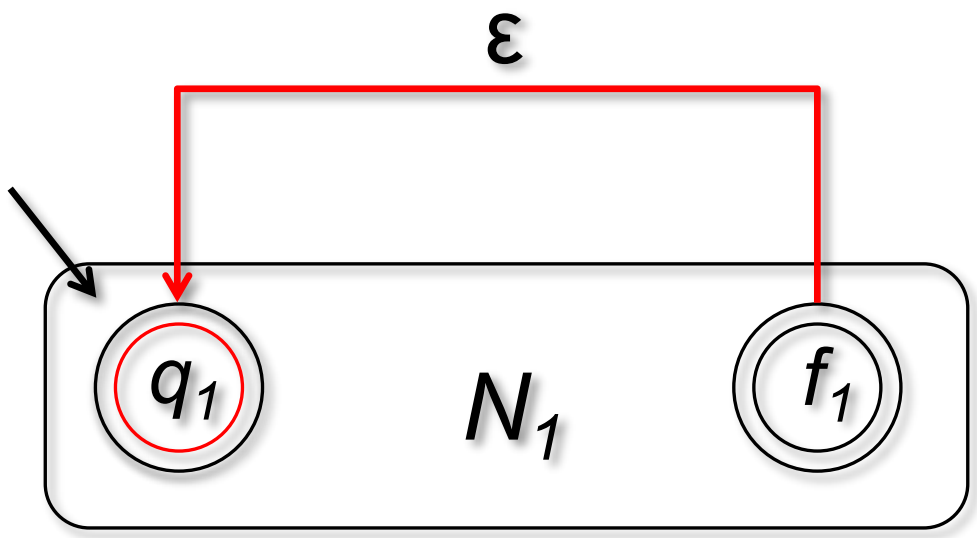

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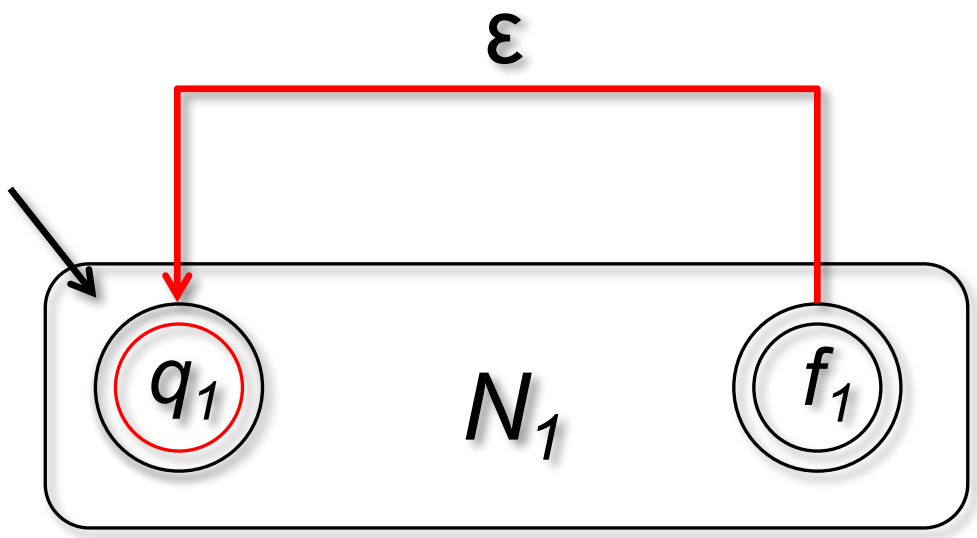

Does not work! Why?

Closure under Kleene star

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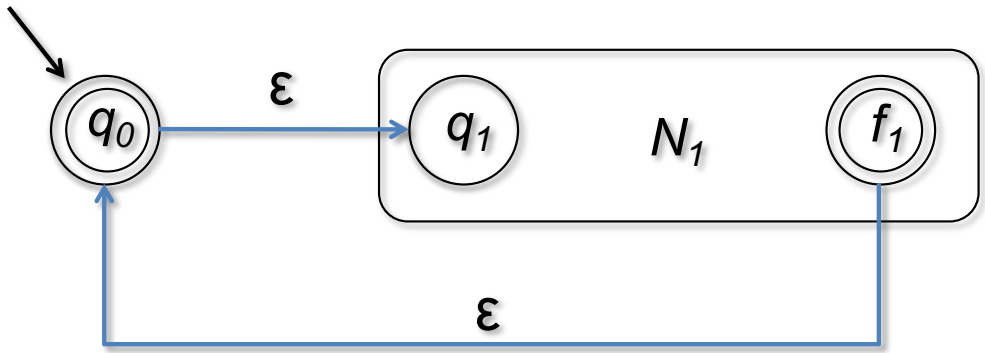

Part II

NFAs capture Regular Languages

Regular Languages Recap

Regular Languages	**Regular Expressions**
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	$\mathbf{a}$ denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1 r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

NFAs and Regular Language

Theorem

For every regular language L there is an NFA N such that $L = L(N)$.

Proof strategy:

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

NFAs and Regular Language

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
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Base cases: $\emptyset$, ϵ, a for $a \in \Sigma$

NFAs and Regular Language

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

Inductive cases:

- r_1, r_2 regular expressions and $r = r_1 + r_2$.

NFAs and Regular Language

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  By induction there are NFAs N_1, N_2 s.t
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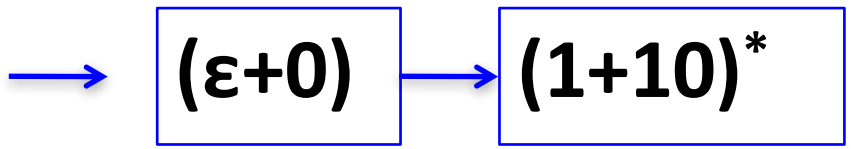
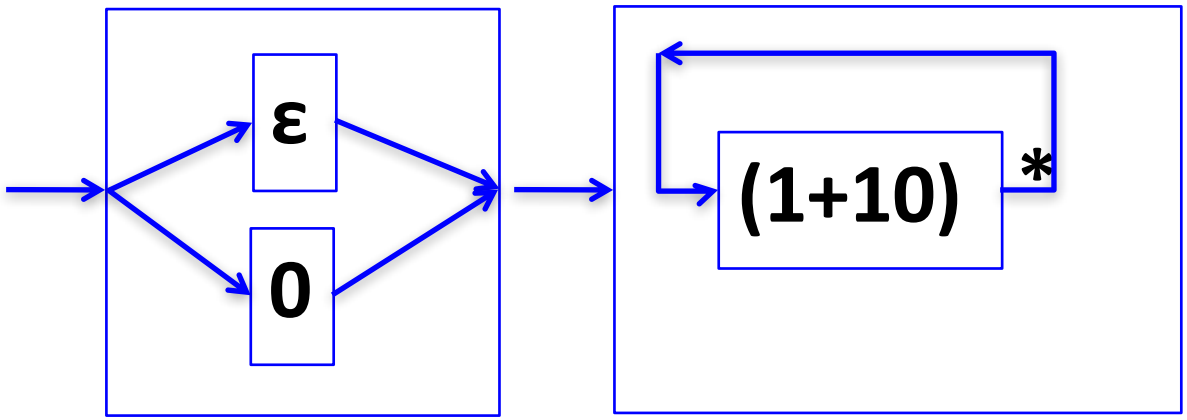
  $L(N) = L(r)$
- $r = r_1 \bullet r_2$. Use closure of NFA languages under concatenation
- $r = (r_1)^*$. Use closure of NFA languages under Kleene star

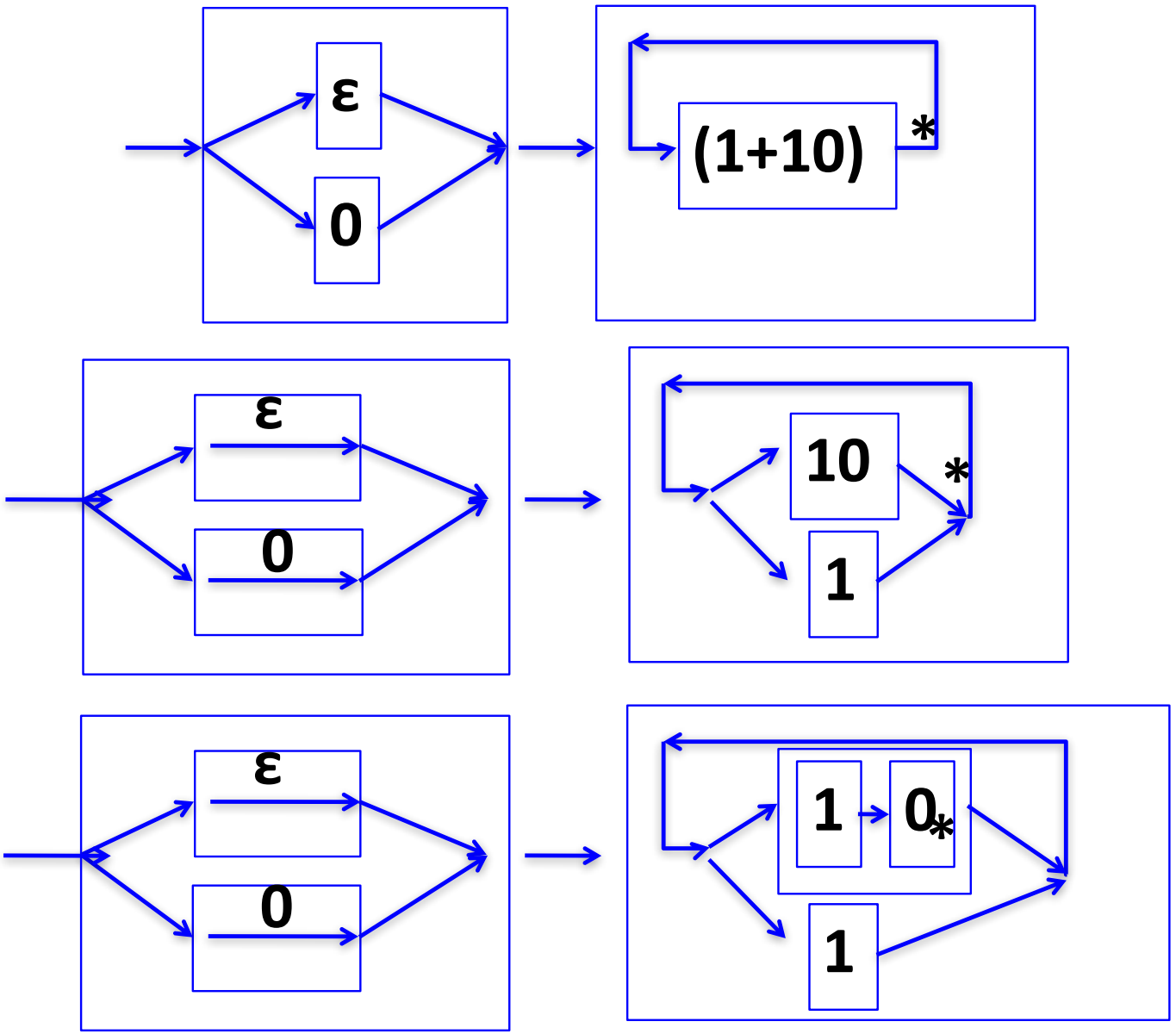
Example

$(\varepsilon+0)(1+10)^*$





Example



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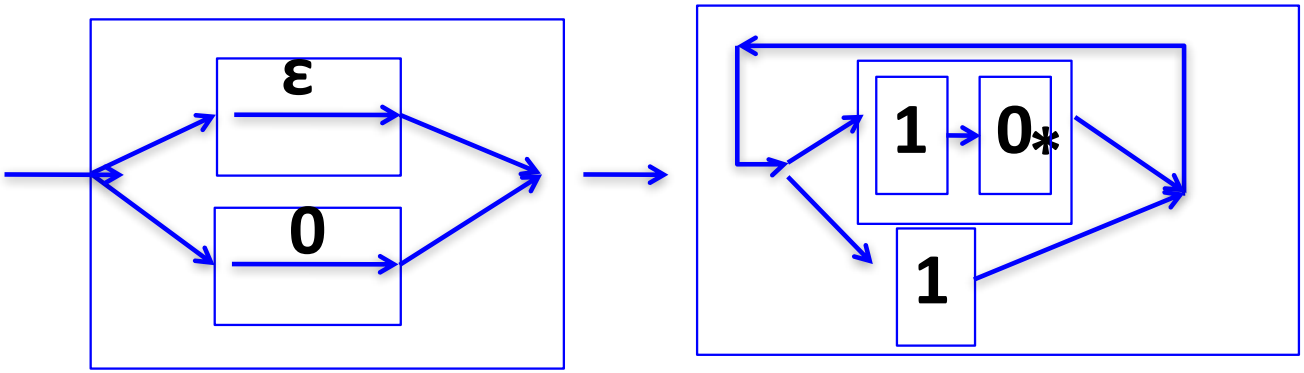

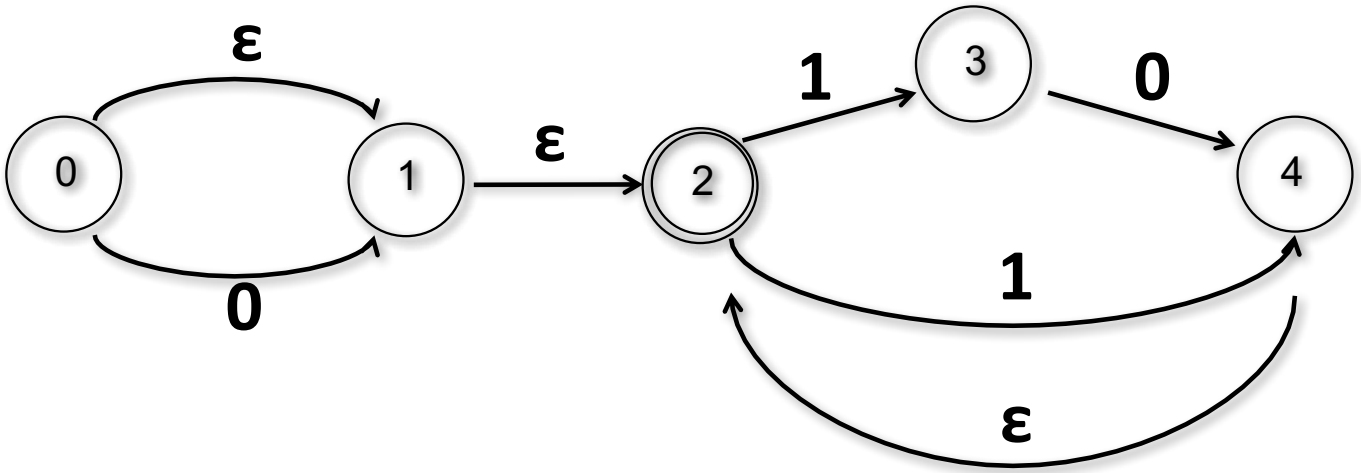

Final NFA simplified slightly to reduce states

Part III

Equivalence of NFAs and DFAs

Equivalence of NFAs and DFAs

Theorem

For every NFA N there is a DFA M such that $L(M) = L(N)$.

Formal Tuple Notation for NFA

Definition

A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- Q is a finite set whose elements are called states,
- Σ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \cup \{\epsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function (here $\mathcal{P}(Q)$ is the power set of Q),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\epsilon\}$ is a susbet of Q — a set of states.

Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ϵ-transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \epsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $\delta^*(q, a) = \cup_{p \in \epsilon\text{reach}(q)} (\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$
- if $w = xa$,
  $\delta^*(q, w) = \cup_{p \in \delta^*(q,x)} (\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$

Formal definition of language accepted by N

Definition

A string w is accepted by NFA N if $\delta_N^*(s, w) \cap A \neq \emptyset$.

Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Simulating an NFA by a DFA

- Think of a program with fixed memory that needs to simulate NFA N on input w.
- What does it need to store after seeing a prefix x of w?

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- Is it sufficient?

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- When should the program accept a string w?

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- When should the program accept a string w? If $\delta^*(s, w) \cap A \neq \emptyset$.

Key Observation: A DFA M that simulates N should keep in its memory/state the set of states of N

Thus the state space of the DFA should be $\mathcal{P}(Q)$.

Subset Construction

NFA $N = (Q, \Sigma, s, \delta, A)$. We create a DFA $M = (Q', \Sigma, \delta', s', A')$ as follows:

- $Q' = \mathcal{P}(Q)$

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- $Q' = \mathcal{P}(Q)$
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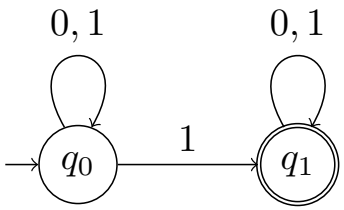
- $A' = \{X \subseteq Q \mid X \cap A \neq \emptyset\}$
- $\delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$ for each $X \subseteq Q$, $a \in \Sigma$.

Example

No ϵ-transitions



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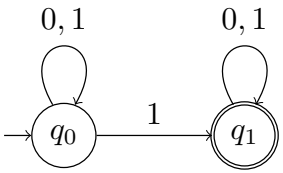
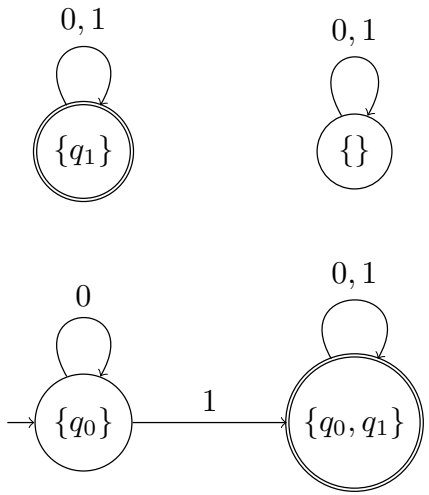
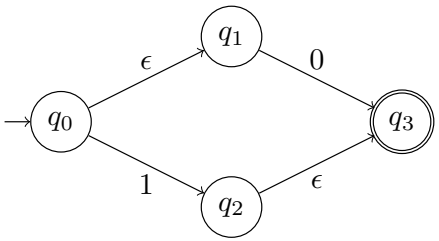
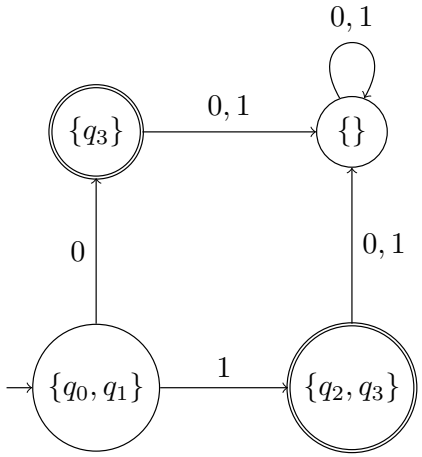




Incremental construction

Only build states reachable from $s' = \epsilon\text{reach}(s)$ the start state of M





$\delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$

Incremental algorithm

- Build M beginning with start state $s' == \epsilon\text{reach}(s)$
- For each existing state $X \subseteq Q$ consider each $a \in \Sigma$ and calculate the state $Y = \delta'(X, a) = \cup_{q \in X} \delta^*(q, a)$ and add a transition.
- If Y is a new state add it to reachable states that need to explored.

To compute $\delta^*(q, a)$ - set of all states reached from q on string a

- Compute $X = \epsilon\text{reach}(q)$
- Compute $Y = \cup_{p \in X} \delta(p, a)$
- Compute $Z = \epsilon\text{reach}(Y) = \cup_{r \in Y} \epsilon\text{reach}(r)$

Proof of Correctness

Theorem

Let $N = (Q, \Sigma, s, \delta, A)$ *be a NFA and let* $M = (Q', \Sigma, \delta', s', A')$ *be a DFA constructed from* N *via the subset construction. Then* $L(N) = L(M)$.

Proof of Correctness

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Let $N = (Q, \Sigma, s, \delta, A)$ be a NFA and let $M = (Q', \Sigma, \delta', s', A')$ be a DFA constructed from N via the subset construction. Then $L(N) = L(M)$.

Stronger claim:

Lemma

For every string w, $\delta^*_N(s, w) = \delta^*_M(s', w)$.

Proof by induction on $|w|$.

Base case: $w = \epsilon$.
$\delta^*_N(s, \epsilon) = \epsilon\text{reach}(s)$.
$\delta^*_M(s', \epsilon) = s' = \epsilon\text{reach}(s)$ by definition of s'.

Proof continued

Lemma

For every string w, $\delta^*_N(s, w) = \delta^*_M(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta^*_N(s, xa) = \cup_{p \in \delta^*_N(s,x)} \delta^*_N(p, a)$ by inductive defn of δ^*_N

Proof continued

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$\delta^*_M(s', xa) = \delta_M(\delta^*_M(s', x), a)$ by inductive defn of δ^*_M

Proof continued

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By inductive hypothesis: $Y = \delta^*_N(s, x) = \delta^*_M(s, x)$

Proof continued

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By inductive hypothesis: $Y = \delta^*_N(s, x) = \delta^*_M(s, x)$

Thus $\delta^*_N(s, xa) = \cup_{p \in Y} \delta^*_N(p, a) = \delta_M(Y, a)$ by definition of δ_M.

Proof continued

Lemma

For every string w, $\delta_N^*(s, w) = \delta_M^*(s', w)$.

Inductive step: $w = xa$ (Note: suffix definition of strings)

$\delta_N^*(s, xa) = \cup_{p \in \delta_N^*(s,x)} \delta_N^*(p, a)$ by inductive defn of δ_N^*

$\delta_M^*(s', xa) = \delta_M(\delta_M^*(s', x), a)$ by inductive defn of δ_M^*

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Therefore,

$\delta_N^*(s, xa) = \delta_M(Y, a) = \delta_M(\delta_M^*(s', x), a) = \delta_M^*(s', xa)$

which is what we need.

Part IV

Closure Properties of Regular Languages

Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

Regular Languages

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- Inductive definition via base cases and closure under union, concatenation and Kleene star
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Regular language closed under many operations:

- union, concatenation, Kleene star via inductive definition or NFAs
- complement, union, intersection via DFAs
- homomorphism, inverse homomorphism, reverse, ...

Different representations allow for flexibility in proofs

Examples: PREFIX and SUFFIX

Let L be a language over Σ.

Definition

$\text{PREFIX}(L) = \{w \mid wx \in L, x \in \Sigma^*\}$

Definition

$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

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Theorem

If L is regular then $PREFIX(L)$ is regular.

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If L is regular then $SUFFIX(L)$ is regular.

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Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes L

Create new DFA/NFA to accept PREFIX(L) (or SUFFIX(L)).

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$X = \{q \in Q \mid s \text{ can reach } q \text{ in } M\}$
$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$
$Z = X \cap Y$

Theorem

Consider DFA $M' = (Q, \Sigma, \delta, s, Z)$. $L(M') = PREFIX(L)$.

SUFFIX

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Consider NFA $N = (Q \cup \{s'\}, \Sigma, \delta', s', A)$. Add new start state s' and ϵ-transition from s' to each state in X.

Claim: $L(N) = \text{SUFFIX}(L)$.

Part V

DFA to Regular Expressions

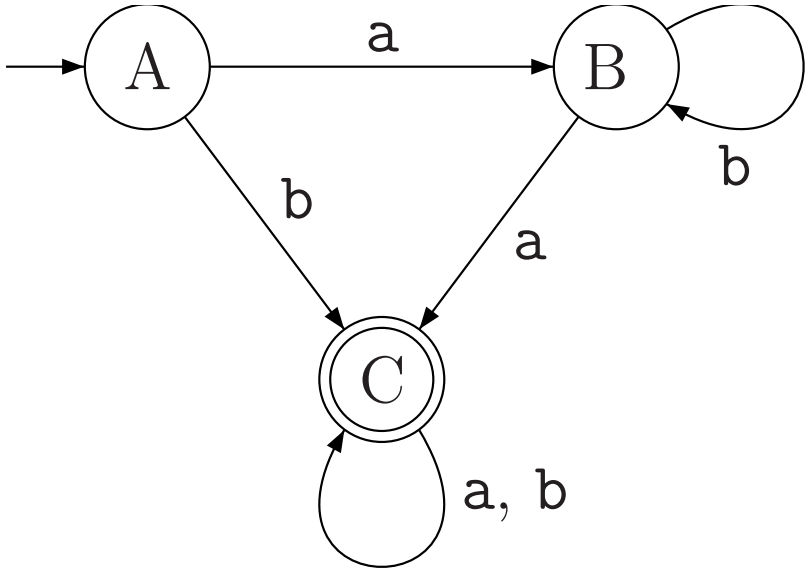
DFA to Regular Expressions

Theorem

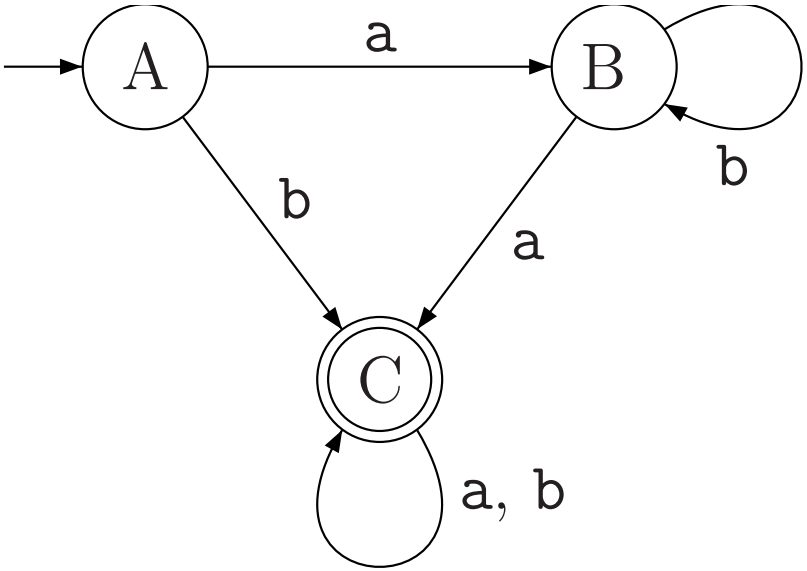
Given a DFA $M = (Q, \Sigma, \delta, s, A)$ there is a regular expression r such that $L(r) = L(M)$. That is, regular expressions are as powerful as DFAs (and hence also NFAs).

- Simple algorithm but formal proof is involved. See notes.
- An easier proof via a more involved algorithm later in course.

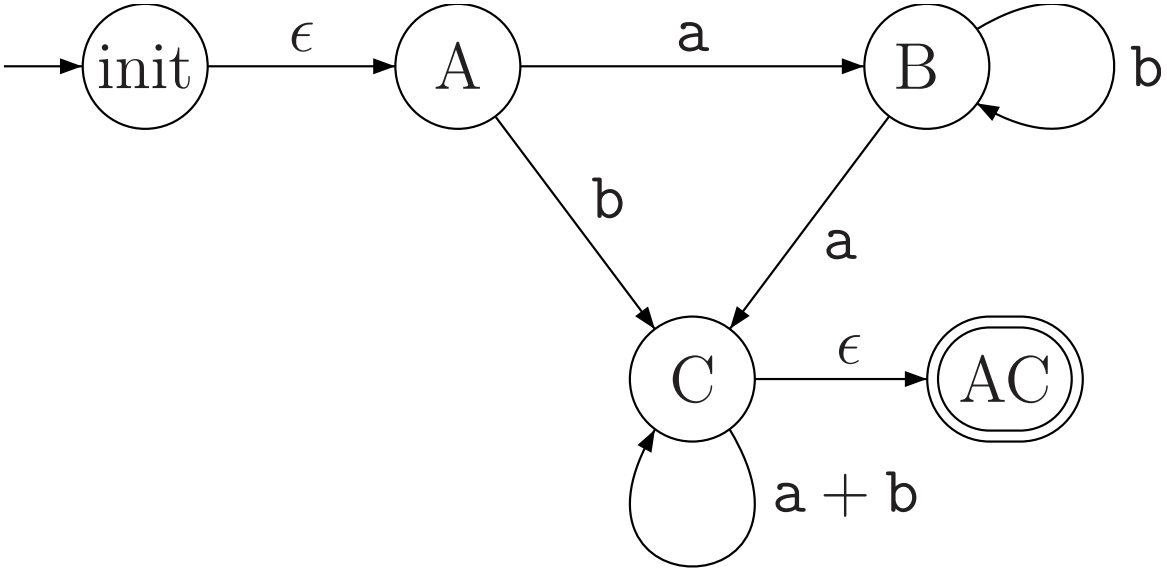
Stage 0: Input



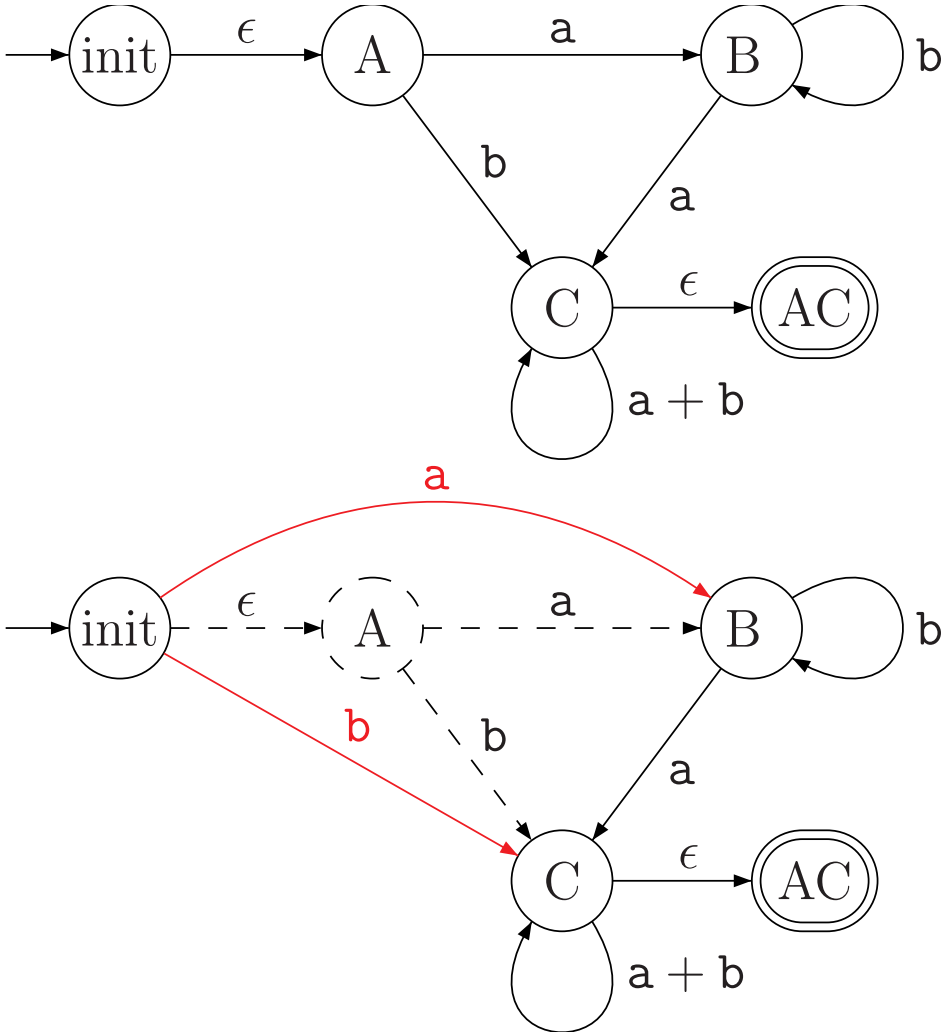
Stage 1: Normalizing



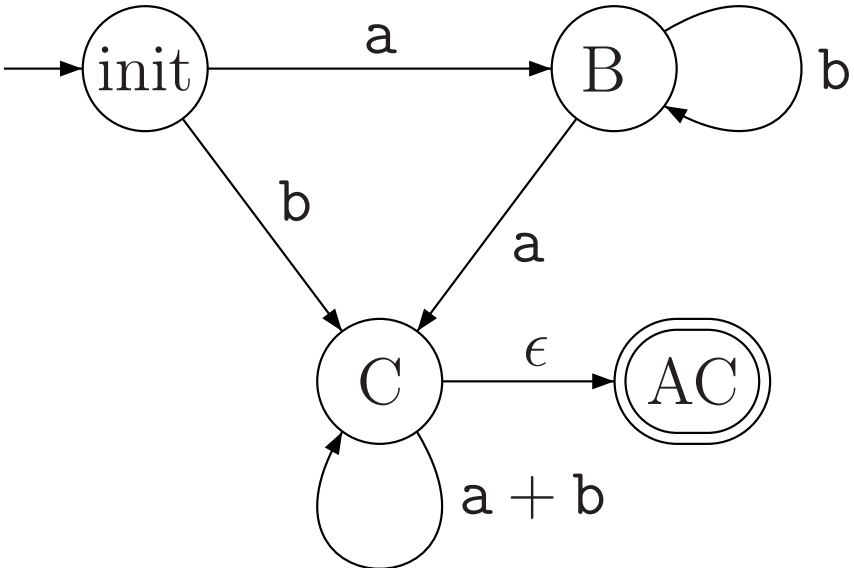
2: Normalizing it.



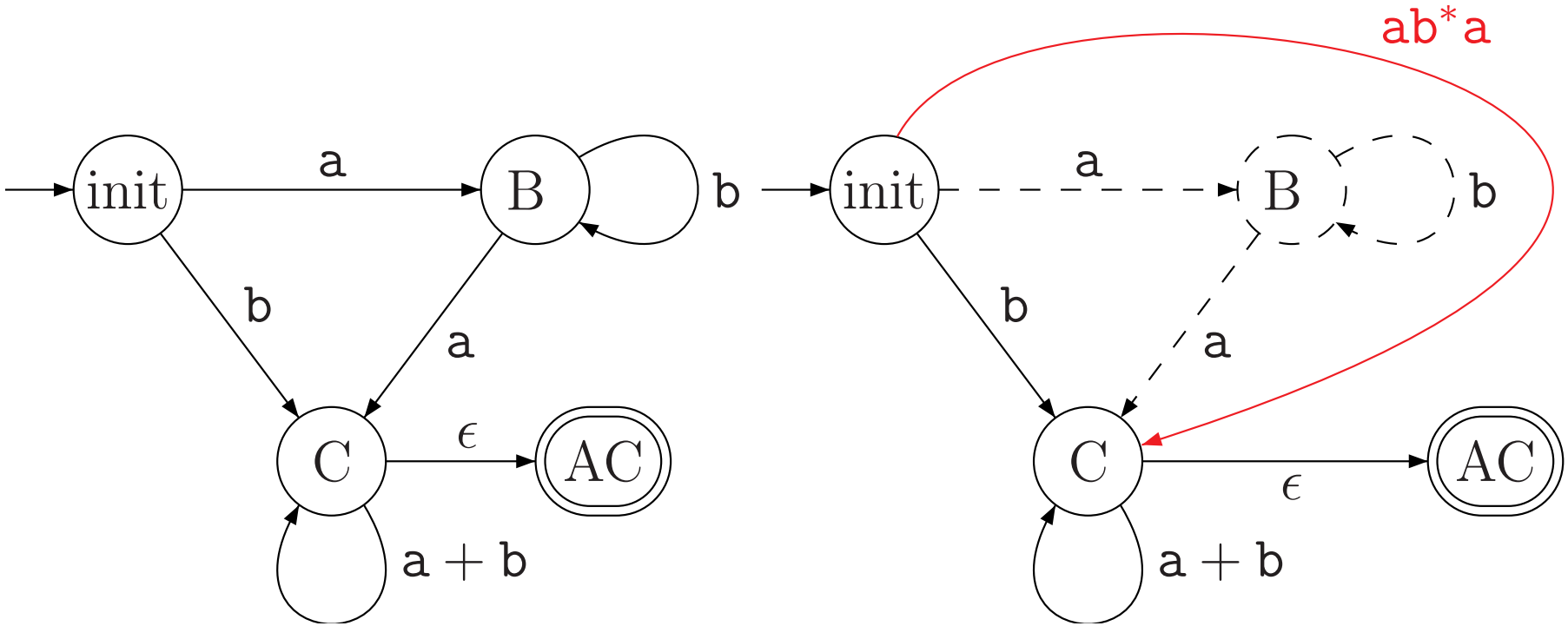
Stage 2: Remove state A



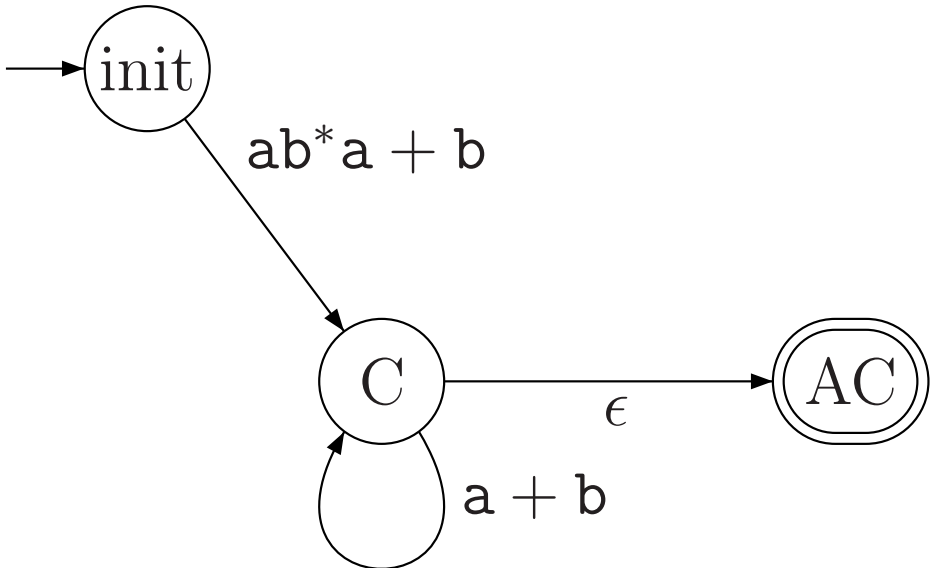
Stage 4: Redrawn without old edges



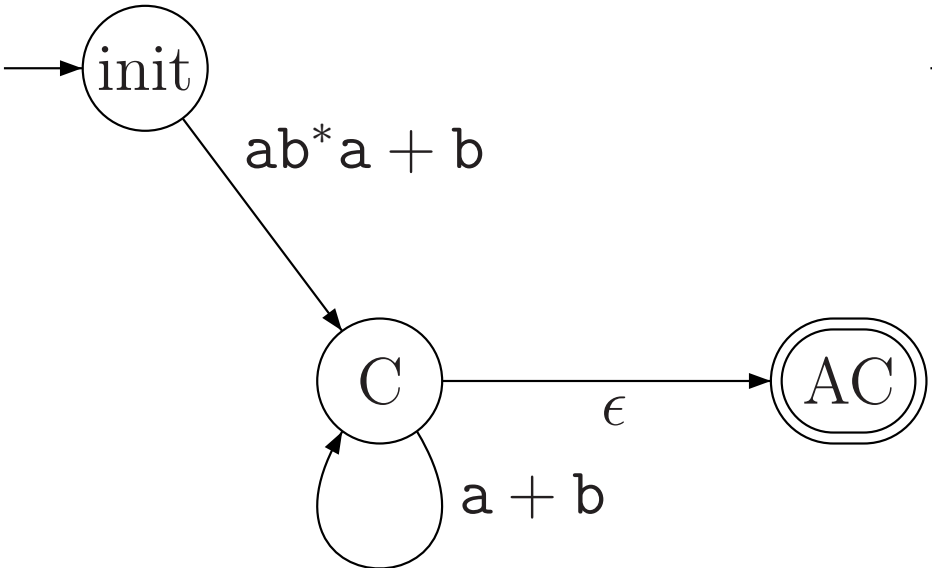
Stage 4: Removing B

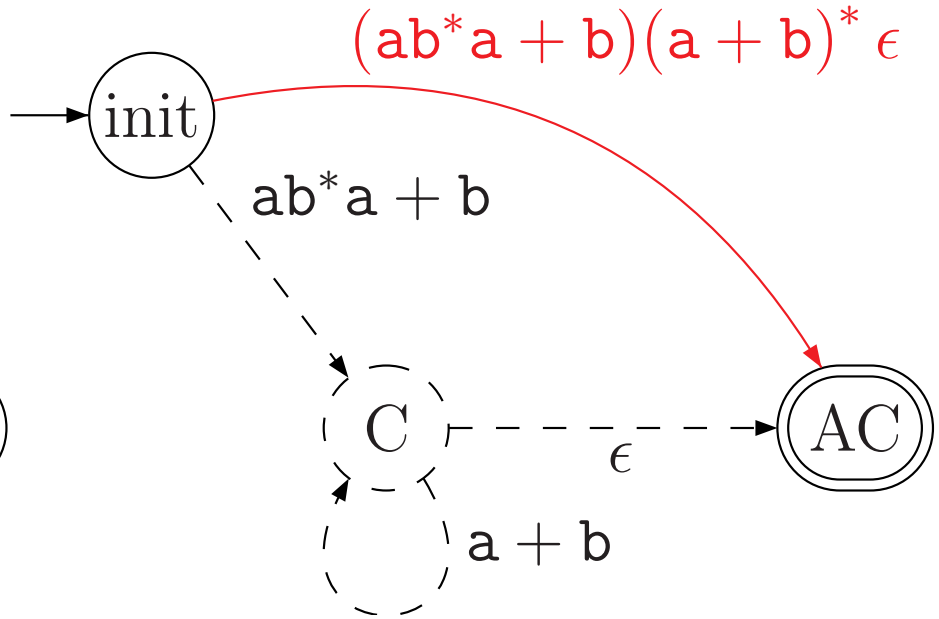


Stage 5: Redraw

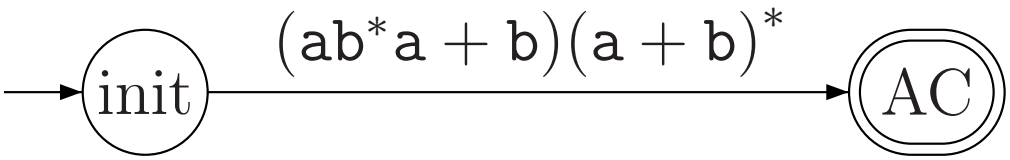


Stage 6: Removing C

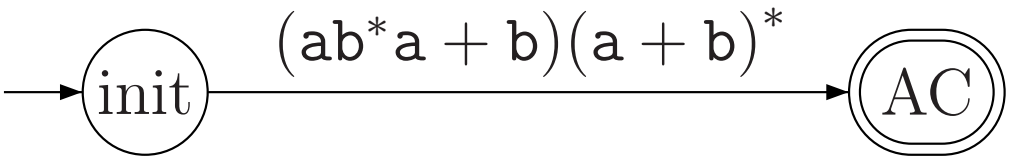




Stage 7: Redraw



Stage 8: Extract regular expression



Thus, this automata is equivalent to the regular expression $(ab^*a + b)(a + b)^*$.