

🌀 Homework 9 🌀

Due Tuesday, November 19, 2019 at 8pm

1. This problem asks you to develop polynomial-time algorithms for two (apparently) minor variants of 3SAT.

- (a) The input to **2SAT** is a boolean formula  $\Phi$  in conjunctive normal form, with exactly **two** literals per clause, and the 2SAT problem asks whether there is an assignment to the variables of  $\Phi$  such that every clause contains at least one TRUE literal.

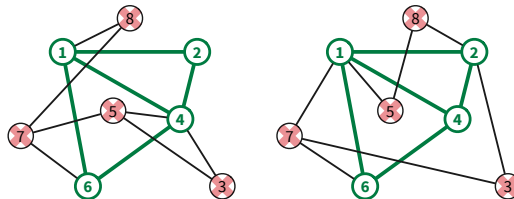
Describe a polynomial-time algorithm for 2SAT. [Hint: This problem is strongly connected to topics covered earlier in the semester.]

- (b) The input to **MAJORITY3SAT** is a boolean formula  $\Phi$  in conjunctive normal form, with exactly three literals per clause. MAJORITY3SAT asks whether there is an assignment to the variables of  $\Phi$  such that every clause contains *at least two* TRUE literals.

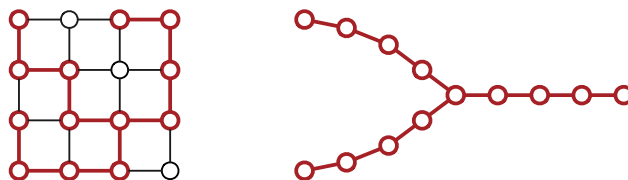
Describe and analyze a polynomial-time reduction from MAJORITY3SAT to 2SAT. Don't forget to prove that your reduction is correct.

- (c) Combining parts (a) and (b) gives us an algorithm for MAJORITY3SAT. What is the running time of this algorithm?

2. Suppose we are given two graphs  $G_1 = (V, E_1)$  and  $G_2 = (V, E_2)$  with the same set of vertices  $V = \{1, 2, \dots, n\}$ . Prove that it is NP-hard to find the smallest subset  $S \subseteq V$  of vertices whose deletion leaves identical subgraphs  $G_1 \setminus S = G_2 \setminus S$ . For example, given the graphs below, the smallest subset has size 4.



3. A **wye** is an undirected graph that looks like the capital letter Y. More formally, a wye consists of three paths of equal length with one common endpoint, called the *hub*.



This grid graph contains a wye whose paths have length 4.

Prove that the following problem is NP-hard: Given an undirected graph  $G$ , what is the largest wye that is a subgraph of  $G$ ? The three paths of the wye must not share any vertices except the hub, and they must have exactly the same length.

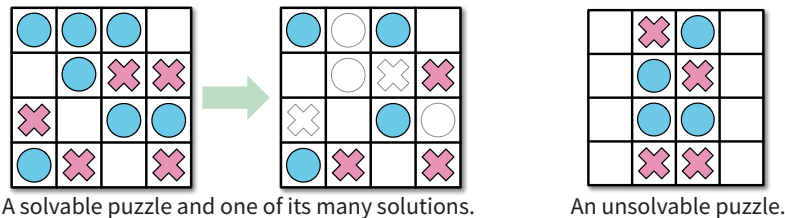
## Solved Problem

4. Consider the following solitaire game. The puzzle consists of an  $n \times m$  grid of squares, where each square may be empty, occupied by a red stone, or occupied by a blue stone. The goal of the puzzle is to remove some of the given stones so that the remaining stones satisfy two conditions:

- (1) Every row contains at least one stone.
- (2) No column contains stones of both colors.

For some initial configurations of stones, reaching this goal is impossible; see the example below.

Prove that it is NP-hard to determine, given an initial configuration of red and blue stones, whether this puzzle can be solved.



**Solution:** We show that this puzzle is NP-hard by describing a reduction from 3SAT.

Let  $\Phi$  be a 3CNF boolean formula with  $m$  variables and  $n$  clauses. We transform this formula into a puzzle configuration in polynomial time as follows. The size of the board is  $n \times m$ . The stones are placed as follows, for all indices  $i$  and  $j$ :

- If the variable  $x_j$  appears in the  $i$ th clause of  $\Phi$ , we place a blue stone at  $(i, j)$ .
- If the negated variable  $\bar{x}_j$  appears in the  $i$ th clause of  $\Phi$ , we place a red stone at  $(i, j)$ .
- Otherwise, we leave cell  $(i, j)$  blank.

**We claim that this puzzle has a solution if and only if  $\Phi$  is satisfiable.** This claim immediately implies that solving the puzzle is NP-hard. We prove our claim as follows:

- $\implies$  First, suppose  $\Phi$  is satisfiable; consider an arbitrary satisfying assignment. For each index  $j$ , remove stones from column  $j$  according to the value assigned to  $x_j$ :
- If  $x_j = \text{TRUE}$ , remove all red stones from column  $j$ .
  - If  $x_j = \text{FALSE}$ , remove all blue stones from column  $j$ .

In other words, remove precisely the stones that correspond to FALSE literals. Because every variable appears in at least one clause, each column now contains stones of only one color (if any). On the other hand, each clause of  $\Phi$  must contain at least one TRUE literal, and thus each row still contains at least one stone. We conclude that the puzzle is satisfiable.

⇐ On the other hand, suppose the puzzle is solvable; consider an arbitrary solution. For each index  $j$ , assign a value to  $x_j$  depending on the colors of stones left in column  $j$ :

- If column  $j$  contains blue stones, set  $x_j = \text{TRUE}$ .
- If column  $j$  contains red stones, set  $x_j = \text{FALSE}$ .
- If column  $j$  is empty, set  $x_j$  arbitrarily.

In other words, assign values to the variables so that the literals corresponding to the remaining stones are all TRUE. Each row still has at least one stone, so each clause of  $\Phi$  contains at least one TRUE literal, so this assignment makes  $\Phi = \text{TRUE}$ . We conclude that  $\Phi$  is satisfiable.

This reduction clearly requires only polynomial time. ■

**Rubric (Standard polynomial-time reduction rubric):** 10 points =

- + 3 points for the reduction itself
  - For an NP-hardness proof, the reduction must be from a known NP-hard problem. You can use any of the NP-hard problems listed in the lecture notes (except the one you are trying to prove NP-hard, of course). **See the list on the next page.**
- + 3 points for the “if” proof of correctness
- + 3 points for the “only if” proof of correctness
- + 1 point for writing “polynomial time”
- An incorrect polynomial-time reduction that still satisfies half of the correctness proof is worth at most 4/10.
- A reduction in the wrong direction is worth 0/10.

**Some useful NP-hard problems.** You are welcome to use any of these in your own NP-hardness proofs, except of course for the specific problem you are trying to prove NP-hard.

**CIRCUITSAT:** Given a boolean circuit, are there any input values that make the circuit output TRUE?

**3SAT:** Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment?

**MAXINDEPENDENTSET:** Given an undirected graph  $G$ , what is the size of the largest subset of vertices in  $G$  that have no edges among them?

**MAXCLIQUE:** Given an undirected graph  $G$ , what is the size of the largest complete subgraph of  $G$ ?

**MINVERTEXCOVER:** Given an undirected graph  $G$ , what is the size of the smallest subset of vertices that touch every edge in  $G$ ?

**MINSETCOVER:** Given a collection of subsets  $S_1, S_2, \dots, S_m$  of a set  $S$ , what is the size of the smallest subcollection whose union is  $S$ ?

**MINHITTINGSET:** Given a collection of subsets  $S_1, S_2, \dots, S_m$  of a set  $S$ , what is the size of the smallest subset of  $S$  that intersects every subset  $S_i$ ?

**3COLOR:** Given an undirected graph  $G$ , can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

**HAMILTONIANPATH:** Given graph  $G$  (either directed or undirected), is there a path in  $G$  that visits every vertex exactly once?

**HAMILTONIANCYCLE:** Given a graph  $G$  (either directed or undirected), is there a cycle in  $G$  that visits every vertex exactly once?

**TRAVELINGSALESMAN:** Given a graph  $G$  (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in  $G$ ?

**LONGESTPATH:** Given a graph  $G$  (either directed or undirected, possibly with weighted edges), what is the length of the longest simple path in  $G$ ?

**STEINERTREE:** Given an undirected graph  $G$  with some of the vertices marked, what is the minimum number of edges in a subtree of  $G$  that contains every marked vertex?

**SUBSETSUM:** Given a set  $X$  of positive integers and an integer  $k$ , does  $X$  have a subset whose elements sum to  $k$ ?

**PARTITION:** Given a set  $X$  of positive integers, can  $X$  be partitioned into two subsets with the same sum?

**3PARTITION:** Given a set  $X$  of  $3n$  positive integers, can  $X$  be partitioned into  $n$  three-element subsets, all with the same sum?

**INTEGERLINEARPROGRAMMING:** Given a matrix  $A \in \mathbb{Z}^{n \times d}$  and two vectors  $b \in \mathbb{Z}^n$  and  $c \in \mathbb{Z}^d$ , compute  $\max\{c \cdot x \mid Ax \leq b, x \geq 0, x \in \mathbb{Z}^d\}$ .

**FEASIBLEILP:** Given a matrix  $A \in \mathbb{Z}^{n \times d}$  and a vector  $b \in \mathbb{Z}^n$ , determine whether the set of feasible integer points  $\max\{x \in \mathbb{Z}^d \mid Ax \leq b, x \geq 0\}$  is empty.

**DRAUGHTS:** Given an  $n \times n$  international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

**SUPERMARIOBROTHERS:** Given an  $n \times n$  Super Mario Brothers level, can Mario reach the castle?

**STEAMEDHAMS:** Aurora borealis? At this time of year, at this time of day, in this part of the country, localized entirely within your kitchen? May I see it?