HW10 out due Tue after break
LAST graded HW

33 HW only count 24

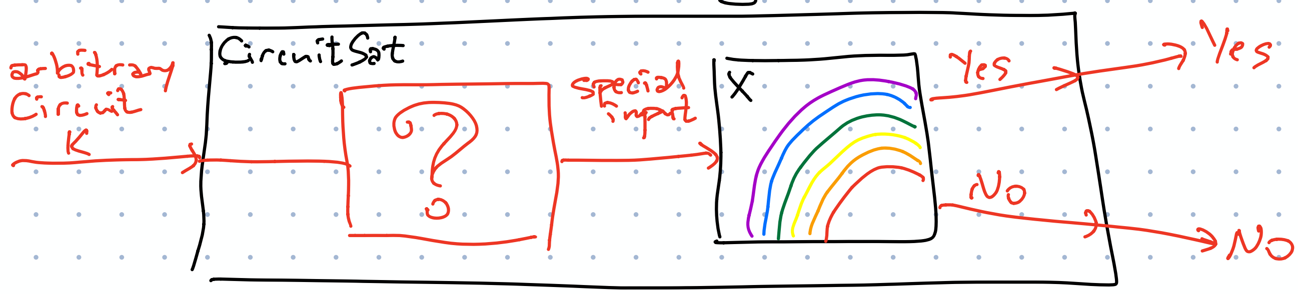
Show problem can't be solved quickly? <u>NP-hard</u>

To prove X is NP-hard:

Poly-time reduction from CircuitSat to X
3SAT
MaxIndSet
MaxClique
MinVertexCover ⟵
3Color
MinColors
4Color
HamCycle
HamPath
TSP
Longest Cycle
Longest Path

- Algorithm transform arbitrary circuit to special instance of X

- Every good circuit ⟶ good instance of X
- Every good instance of X produced by transformation comes from a good circuit



$$T_{CircuitSat}(n) \le O(n) + T_X(n)$$

$$\Longleftrightarrow \quad T_X(n) \ge T_{CircuitSAT}(n) - O(n)$$

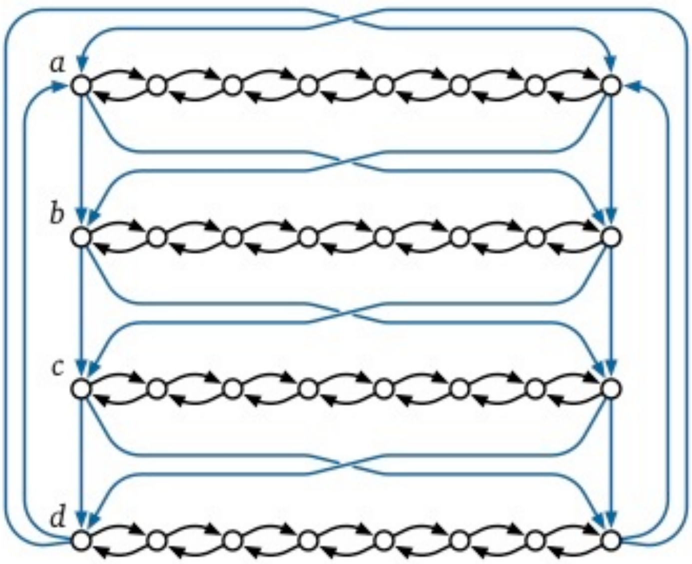
<u>Hamiltonian Cycle</u> Input: Directed graph $G=(V,E)$

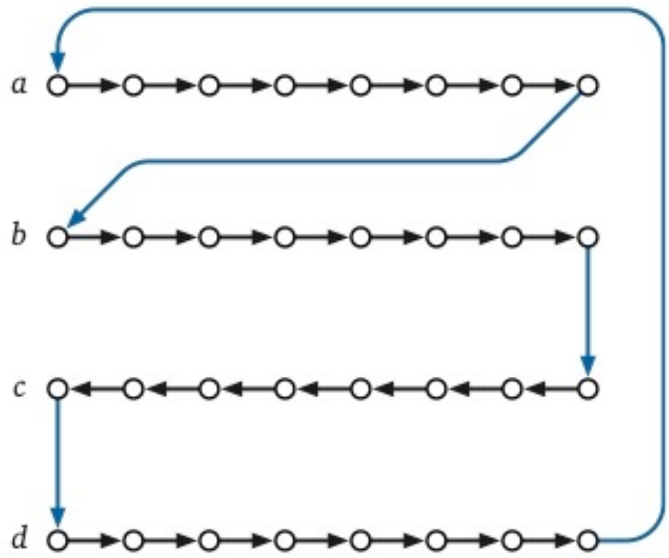
Question: Is there a simple cycle that visits every vertex?

Reduction from 3SAT

Given arbitrary 3CNF formula Φ — n variables
m clauses

① Variable "gadgets"



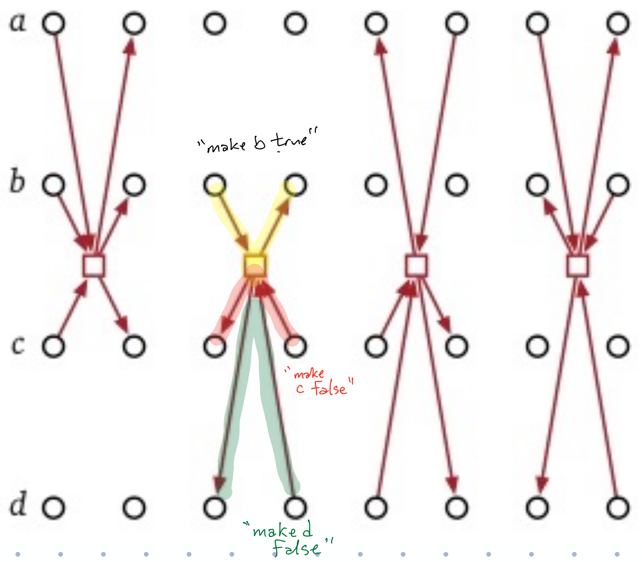


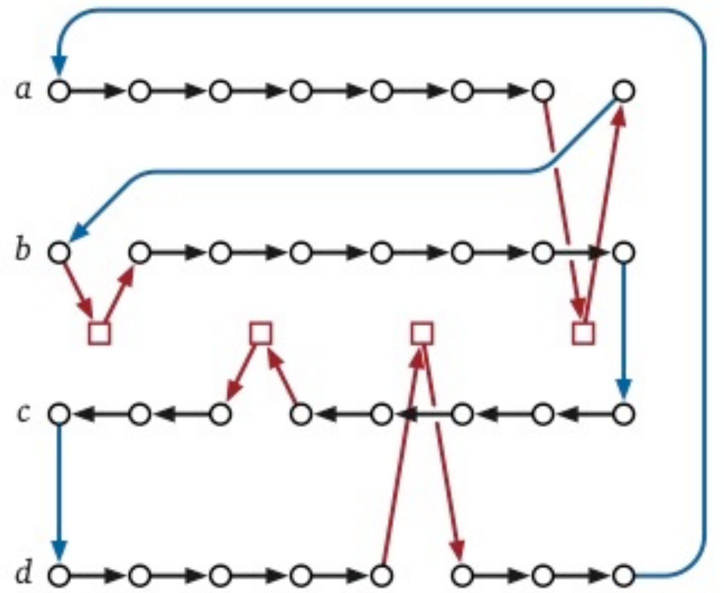
For each variable
"double chain" of $2m$ vertices

Endpoints connected
in cyclic sequence

$a = T$
$b = T$
$c = F$
$d = T$

② Add 1 vertex and 6 edges for each clause





$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$.

$a = T$
$b = T$
$c = F$
$d = T$

satistfying assignment in Φ $\Longleftrightarrow$ Ham cycle in G

poly time by brute force ✓

Some useful NP-hard problems. You are welcome to use any of these in your own NP-hardness proofs, except of course for the specific problem you are trying to prove NP-hard.

CircuitSat: Given a boolean circuit, are there any input values that make the circuit output True?

3Sat: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment?

MaxIndependentSet: Given an undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

MaxClique: Given an undirected graph G, what is the size of the largest complete subgraph of G?

MinVertexCover: Given an undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

MinSetCover: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subcollection whose union is S?

MinHittingSet: Given a collection of subsets $S_1, S_2, \ldots, S_m$ of a set S, what is the size of the smallest subset of S that intersects every subset S_i?

3Color: Given an undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

HamiltonianPath: Given graph G (either directed or undirected), is there a path in G that visits every vertex exactly once?

HamiltonianCycle: Given a graph G (either directed or undirected), is there a cycle in G that visits every vertex exactly once?

TravelingSalesman: Given a graph G (either directed or undirected) with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

LongestPath: Given a graph G (either directed or undirected, possibly with weighted edges), what is the length of the longest simple path in G?

SteinerTree: Given an undirected graph G with some of the vertices marked, what is the minimum number of edges in a subtree of G that contains every marked vertex?

SubsetSum: Given a set X of positive integers and an integer k, does X have a subset whose elements sum to k?

Partition: Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3Partition: Given a set X of $3n$ positive integers, can X be partitioned into n three-element subsets, all with the same sum?

IntegerLinearProgramming: Given a matrix $A \in \mathbb{Z}^{n \times d}$ and two vectors $b \in \mathbb{Z}^n$ and $c \in Z^d$, compute $\max\{c \cdot x \mid Ax \leq b, x \geq 0, x \in \mathbb{Z}^d\}$.

FeasibleILP: Given a matrix $A \in \mathbb{Z}^{n \times d}$ and a vector $b \in \mathbb{Z}^n$, determine whether the set of feasible integer points $\max\{x \in \mathbb{Z}^d \mid Ax \leq b, x \geq 0\}$ is empty.

Draughts: Given an $n \times n$ international draughts configuration, what is the largest number of pieces that can (and therefore must) be captured in a single move?

SuperMarioBrothers: Given an $n \times n$ Super Mario Brothers level, can Mario reach the castle?

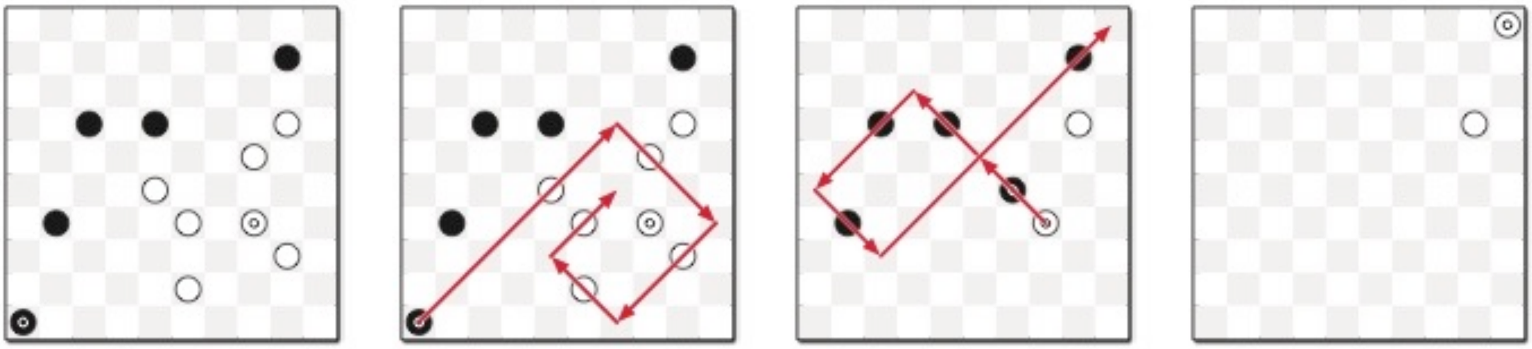
SteamedHams: Aurora borealis? At this time of year, at this time of day, in this part of the country, localized entirely within your kitchen? May I see it?

- Smell test

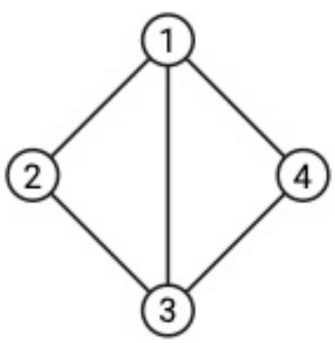
— Choose a subset Partition into 2 subsets Assign 0's and 1's	SAT
— Partition into more than 2 subsets, min # subsets	COLOR
— Largest possible subset	MaxClique
— Smallest possible subset	Min VertexCover
— Ordering objects long sequences	HamiltonianPath/Cycle TSP
— The problem has a 3 in it. — 3PARTITION — 3D Matching	3SAT or 3COLOR
— If all else fails	3SAT CircuitSAT

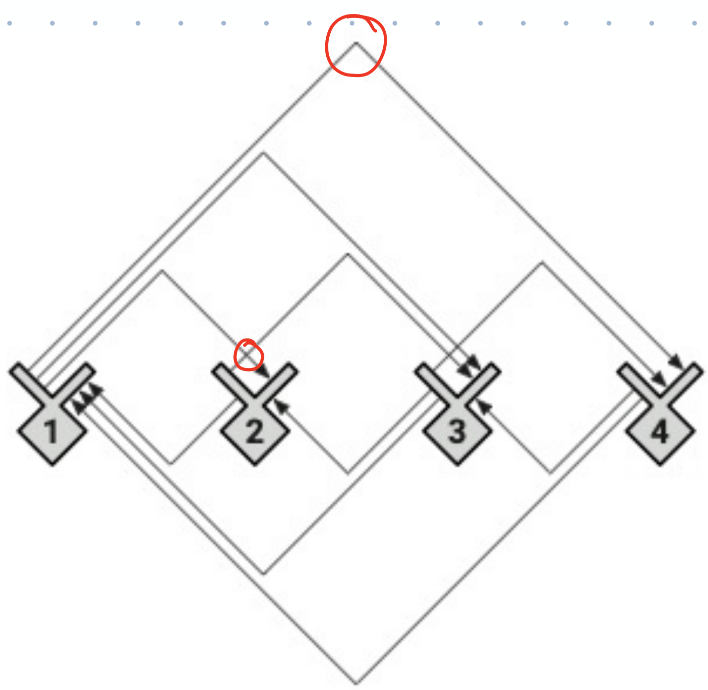
International Draughts ("Checkers")

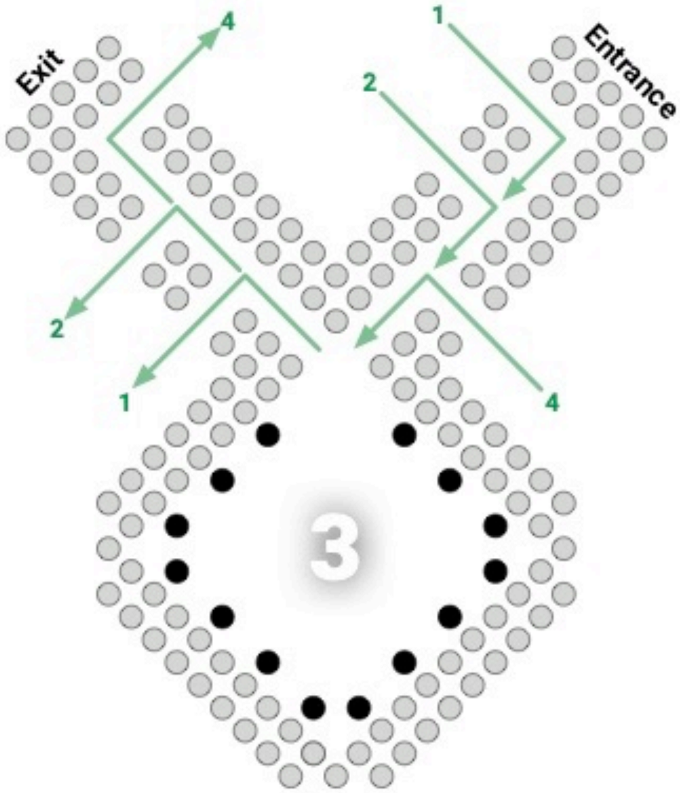
- 10x10 board (n x n variant)
- "Flying kings"
- captured pieces stay until move is over
- forced <u>maximum</u> capture

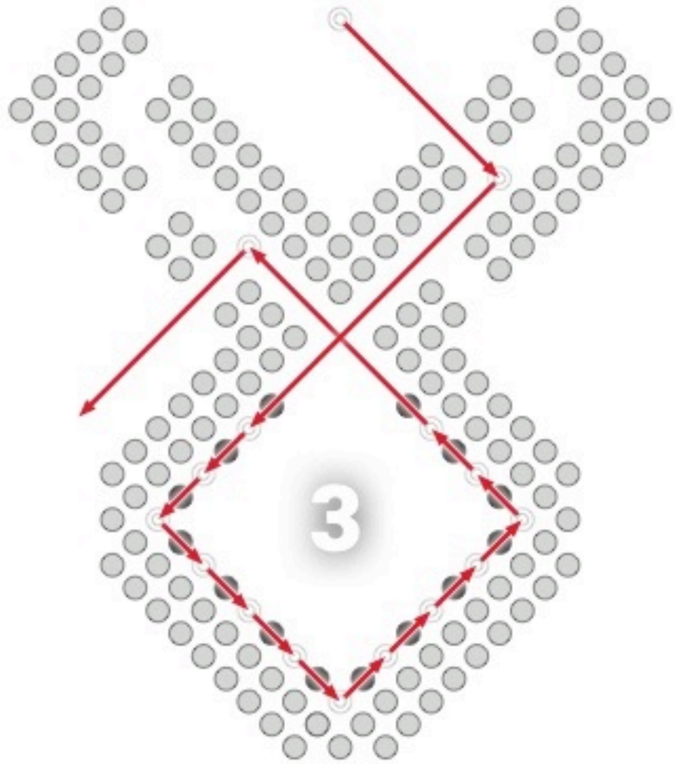


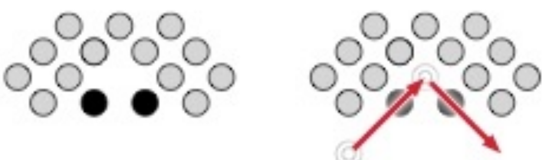
Following these rules is NP-hard

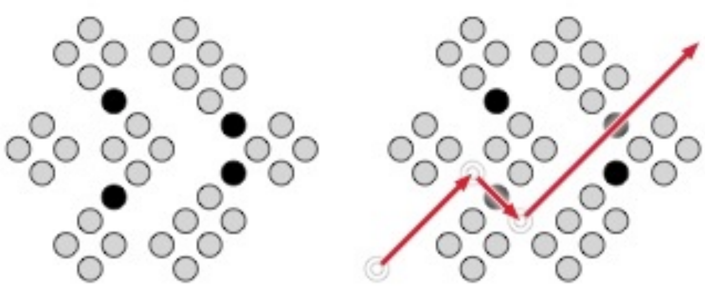


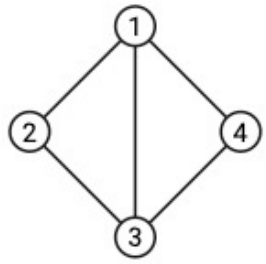



Exit
Entrance
1
2
4
2
1
4
3


3



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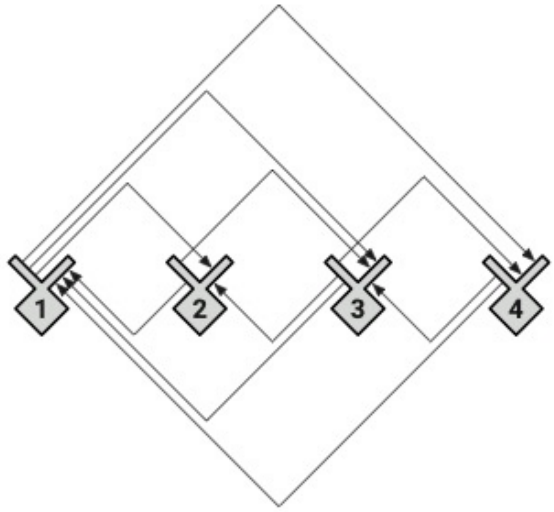
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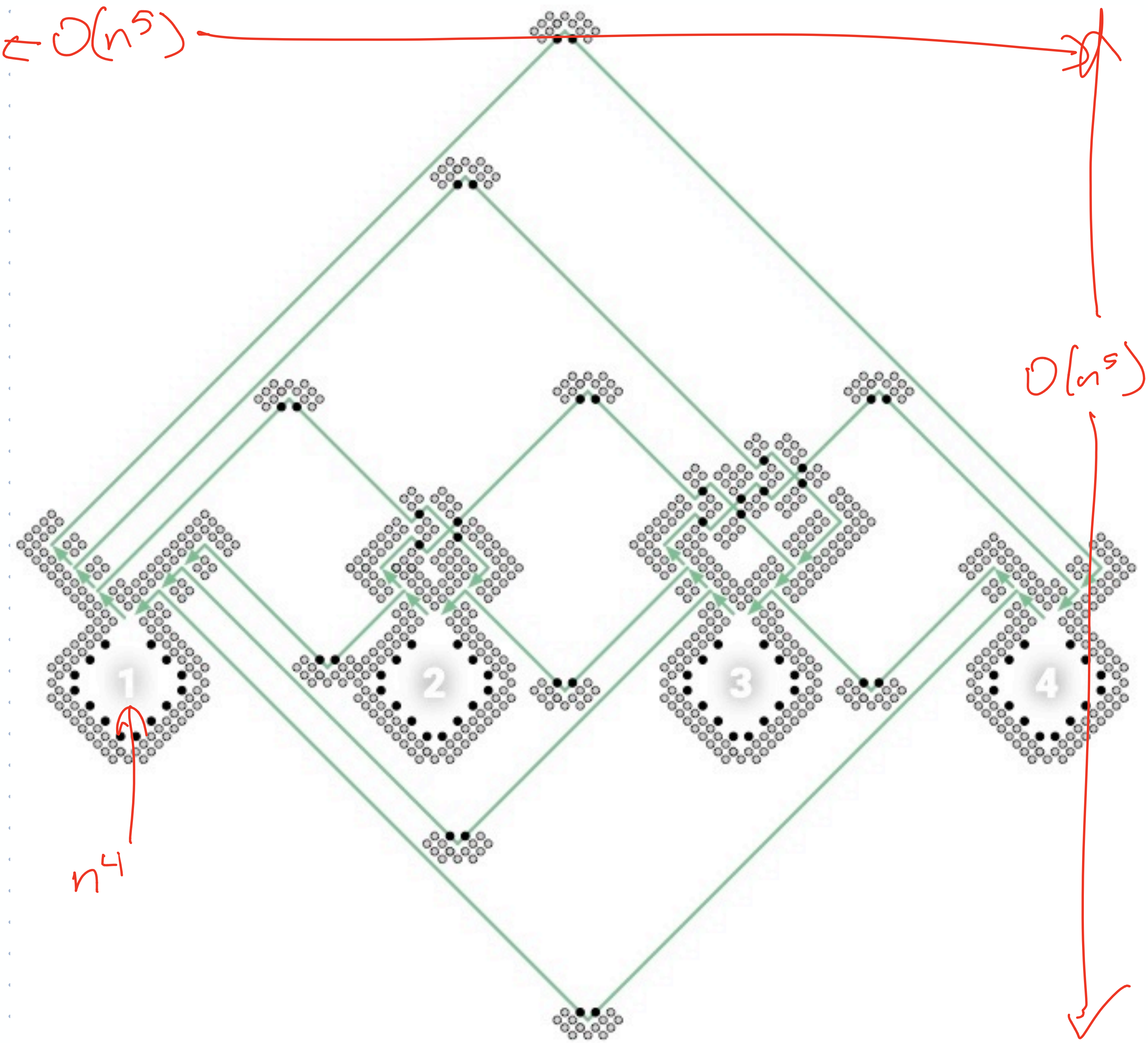
← O(n^5)
O(n^5)
n^4
1
2
3
4