DFA to Regular Expressions, Language Transformations

Lecture 7

September 17, 2019

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

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Languages accepted by DFAs, NFAs, and regular expressions are the same.

- DFAs are special cases of NFAs (trivial)
- NFAs accept regular expressions (we saw already, relative easy)
- DFAs accept languages accepted by NFAs (subset construction)
- Regular expressions for languages accepted by DFAs (sketch today, again later in the course)

Part I

DFA to Regular Expressions

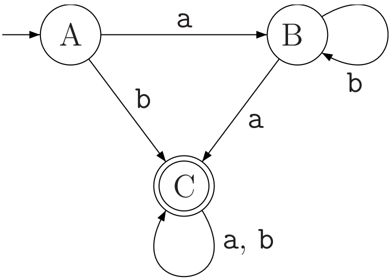
DFA to Regular Expressions

Theorem

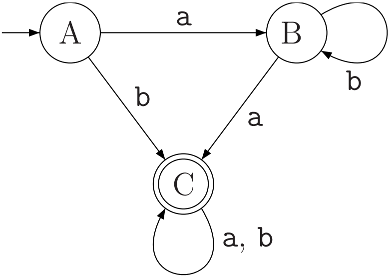
Given a DFA $M = (Q, \Sigma, \delta, s, A)$ *there is a regular expression* r *such that* $L(r) = L(M)$. *That is, regular expressions are as powerful as DFAs (and hence also NFAs).*

- Simple algorithm but formal proof is technical. See notes.
- A different algorithm with an easier formal proof later in the course.

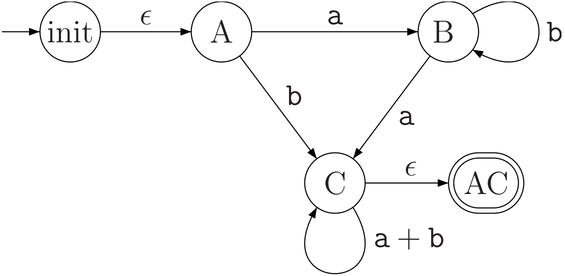
Stage 0: Input



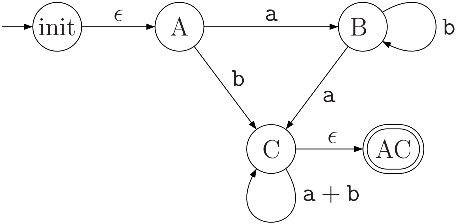
Stage 1: Normalizing

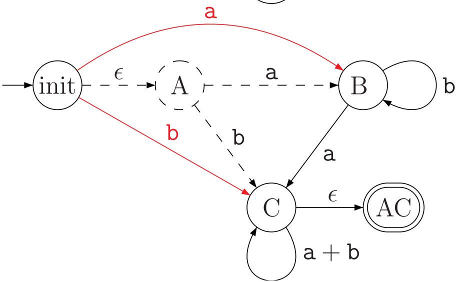


2: Normalizing it.

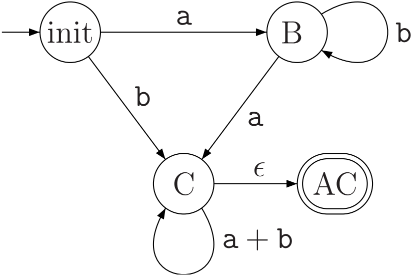


Stage 2: Remove state A

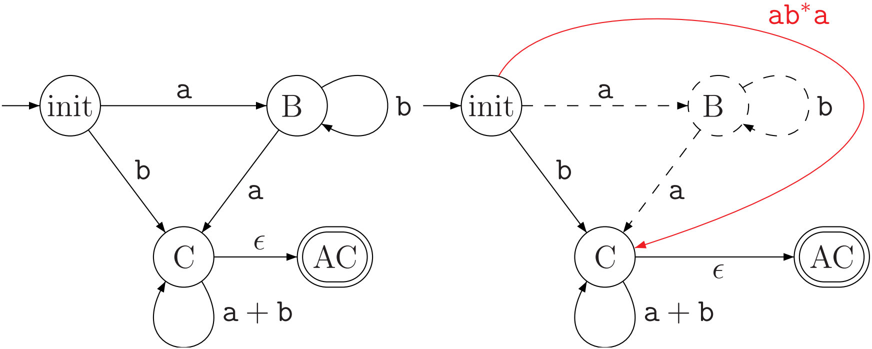




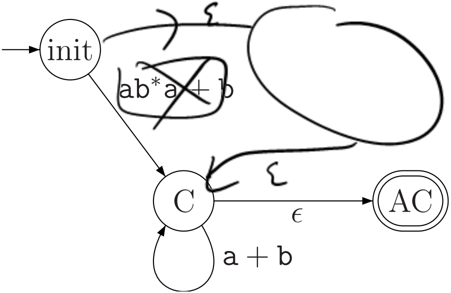
Stage 4: Redrawn without old edges



Stage 4: Removing B



Stage 5: Redraw



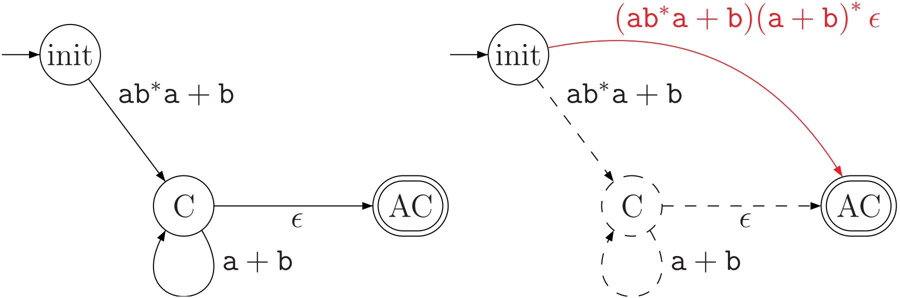
abbba

w

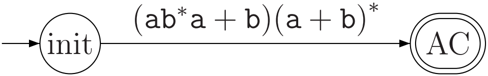
$$s \xrightarrow{\substack{w_1 \\ a_1}} q_1 \xrightarrow{\substack{w_2 \\ a_2}} q_2 \longrightarrow \cdots \xrightarrow{\substack{w_k \\ a_k}} q_f$$

$$w = a_1 a_2 \cdots a_k \qquad q_f \in A$$

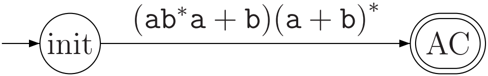
Stage 6: Removing C



Stage 7: Redraw



Stage 8: Extract regular expression



Thus, the automata is equivalent to the regular expression $(ab^*a + b)(a + b)^*$.

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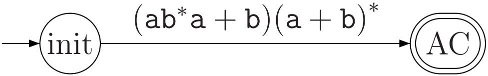

Thus, the automata is equivalent to the regular expression $(ab^*a + b)(a + b)^*$.

- States can be eliminated in any order
- Can start with NFA

Part II

Closure Properties and Language Transformations

Closure propeties

Definition

(Informal) A set A is **closed** under an operation **op** if applying **op** to any elements of A results in an element that also belongs to A.

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(Informal) A set A is **closed** under an operation **op** if applying **op** to any elements of A results in an element that also belongs to A.

Examples:

- *Integers:* closed under $+$, $-$, $*$, but not division.
- *Positive integers:* closed under $+$ but not under $-$
- *Regular languages:* closed under union, intersection, Kleene star, complement, difference, homomorphism, inverse homomorphism, reverse, $\ldots$

Closure properties of Regular Languages

Question: How do we prove that regular languages are closed under some new operation?

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- Use existing closure properties

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  - L_1, L_2, L_3, L_4 regular implies $(L_1 - L_2) \cap (\bar{L_3} \cup L_4)^*$ is regular

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- Transform regular expressions

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Three broad approaches

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  - L_1, L_2, L_3, L_4 regular implies $(L_1 - L_2) \cap (\bar{L_3} \cup L_4)^*$ is regular
- Transform regular expressions
- Transform DFAs to NFAs — versatile technique and shows the power of nondeterminism

Example: REVERSE

Given string w, w^R is reverse of w.
For a language L define $L^R = \{w^R \mid w \in L\}$ as reverse of L.

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Infinitely many regular languages!

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Infinitely many regular languages!

Proof technique:

- take some finite representation of L such as regular expression r
- Describe an algorithm A that takes r as input and outputs a regular expression r' such that $L(r') = (L(r))^R$.
- Come up with A and prove its correctness.

REVERSE via regular expressions

Suppose r is a regular expression for L. How do we create a regular expression r' for L^R?

$(010 + 101)^* 110$

$100(11 + 010)$

$(11 + 010)001$

100

$100 + 0100$

$(100)^* \quad (001)^*$

REVERSE via regular expressions

Suppose r is a regular expression for L. How do we create a regular expression r' for L^R? Inductively based on recursive definition of r.

- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
- $r = r_1 + r_2$
- $r = r_1 \cdot r_2$
- $r = (r_1)^*$

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- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
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If u, v are strings
then $(uv)^R = v^R u^R$

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- $r = (r_1)^*$. If r_1' is reg expressions for $(L(r_1))^R$ then $r' = (r_1')^*$

$$r_1^* = (\epsilon + r_1 + r_1 r_1 + r_1 r_1 r_1 + \cdots)$$
$$\epsilon + r_1' + r_1' r_1' + r_1' r_1' r_1' + \cdots$$
$$= (r_1')^*$$

REVERSE via regular expressions

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$r = (0 + 10)^*(001 + 01)1$ then $r' =$

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- $r = (r_1)^*$. If r'_1 is reg expressions for $(L(r_1))^R$ then $r' = (r'_1)^*$

$r = (0 + 10)^*(001 + 01)1$ then $r' = 1(100 + 10)(0 + 01)^*$

Proof for each identity: tedious case analysis based on definitions of union, concatenation, Kleene star and reverse.

REVERSE via machine transformation

Given DFA $M = (Q, \Sigma, \delta, s, A)$ want NFA N such that $L(N) = (L(M))^R$.

N should accept w^R **iff** M accepts w

M accepts w iff $\delta^*_M(s, w) \in A$

Idea: N reverses transitions of M and starts at a final state of M.

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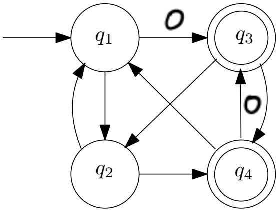
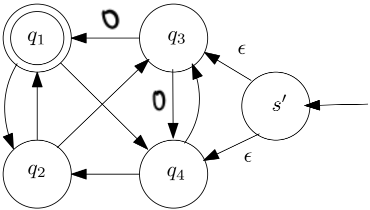
Idea: N reverses transitions of M and starts at a final state of M. Which one? Non-deterministically guesses and accepts if it reaches s.

REVERSE via machine transformation





Caveat: Reversing transitions may create an NFA.

REVERSE via machine transformation

Proof (DFA to NFA): Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L. We construct an NFA $M^R = (\Sigma, Q^R, s^R, A^R, \delta^R)$ with ε-transitions that accepts L^R, intuitively by reversing every transition in M, and swapping the roles of the start state and the accepting states. Because M does not have a unique accepting state, we need to introduce a special start state s^R, with ε-transitions to each accepting state in M. These are the only ε-transitions in M^R.

$$
\begin{aligned}
Q^R &= Q \cup \{s^R\} \\
A^R &= \{s\} \\
\delta^R(s^R, \varepsilon) &= A \\
\delta^R(s^R, a) &= \varnothing && \text{for all } a \in \Sigma \\
\delta^R(q, \varepsilon) &= \varnothing && \text{for all } q \in Q \\
\delta^R(q, a) &= \{p \mid q \in \delta(p, a)\} && \text{for all } q \in Q \text{ and } a \in \Sigma
\end{aligned}
$$

Routine inductive definition-chasing now implies that the reversal of any sequence $q_0 \to q_1 \to \cdots \to q_\ell$ of transitions in M is a valid sequence $q_\ell \to q_{\ell-1} \to \cdots \to q_0$ of transitions in M^R. Because the transitions retain their labels (but reverse directions), it follows that M accepts any string w if and only if M^R accepts w^R.

We conclude that the NFA M^R accepts L^R, so L^R must be regular. $\square$

A more complicated example: CYCLE

$CYCLE(L) = \{yx \mid x, y \in \Sigma^*, xy \in L\}$

Theorem

$CYCLE(L)$ *is regular if* L *is regular.*

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$$CYCLE(L) = \{yx \mid x, y \in \Sigma^*, xy \in L\}$$

Theorem

$CYCLE(L)$ is regular if L is regular.

Example: $L = \{abc, 374a\}$

$CYCLE(L) = \{abc, bca, cab, a374, 4a37, 74a3, 374a\}$

A more complicated example: CYCLE

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Theorem

$CYCLE(L)$ *is regular if* L *is regular.*

Given DFA M for L create NFA N that accepts $CYCLE(L)$.

- N is a finite state machine, cannot know split of w into xy and yet has to simulate M on x and y.
- Exploit fact that M is itself a finite state machine. N only needs to "know" the state $\delta_M^*(s, x)$ and there are only finite number of states in M

Construction for CYCLE

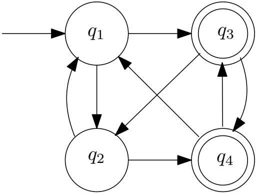
Let $w = xy$ and $w' = yx$.

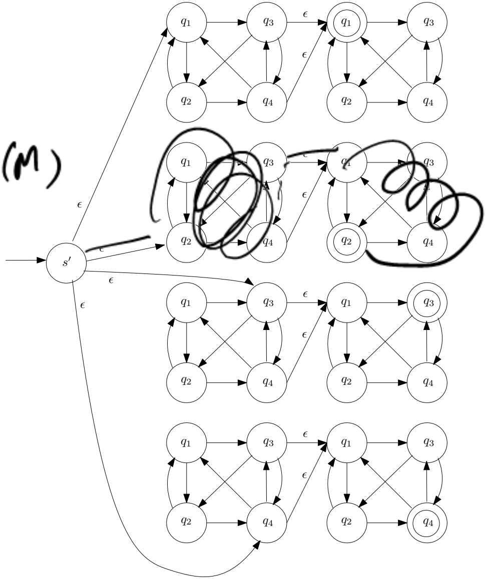
- N *guesses* state $q = \delta^*_M(s, x)$ and simulates M on w' with start state q.
- N *guesses* when y ends (at that point M must be in an accept state) and transitions to a copy of M to simulate M on remaining part of w' (which is x)
- N accepts w' if after second copy of M on x it ends up in the guessed state q

Construction for CYCLE

$w = xy$

$yx \in L(M)$





Proving correctness

Exercise: Write down formal description of N in tuple notation starting with $M = (Q, \Sigma, \delta, s, A)$.

Need to argue that $L(N) = CYCLE(L(M))$

- If $w = xy$ accepted by M then argue that yx is accepted by N
- If N accepts w' then argue that $w' = yx$ such that xy accepted by M.

Application of closure properties to non-regularity

$L_1 = \{0^n 1^n \mid n \geq 0\}$
$L_2 = \{w \in \{0,1\}^* \mid \#_0(w) = \#_1(w)\}$
$L_3 = \{0^i 1^j \mid i \neq j\}$

Application of closure properties to non-regularity

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L_1 is not regular. Can we use that fact to prove L_2 and L_2 are not regular without going through the fooling set argument?

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$L_1 = L_2 \cap 0^* 1^*$ hence if L_2 is regular then L_1 is regular, a contradiction.

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$L_1 = \bar{L_3} \cap 0^*1^*$ hence if L_3 is regular then L_1 is regular, a contradiction

Jeff's reminder about exam

Following topics *not* on the upcoming midterm exam

- Transforming DFA/NFA into regular expressions (covered today)
- Minimizing DFA