CS/ECE 374 ✧ Fall 2019

Homework 2

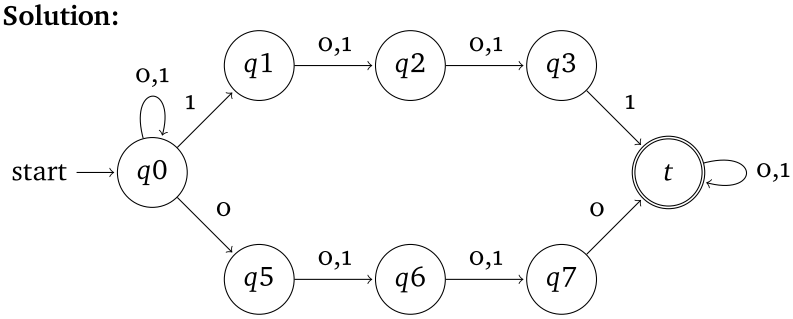
Solutions

---

1. **Building NFAs.** For each of the languages below, construct an NFA that accepts the language. You may either draw the NFA or write out a formal transition function. In either case, you need to label/explain your states and briefly argue why the NFA accepts the correct language.

   (a) All strings over $\Sigma = \{0,1\}$ that have two of the same characters at a distance 3 from each other. E.g., 1011011, 10100.

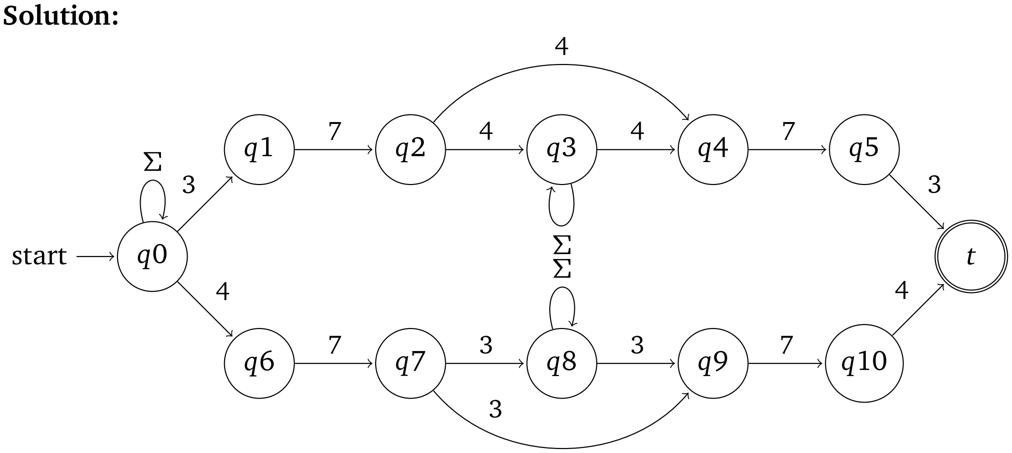
   Solution:



   $q0$: Start state; we have not read the first matching character
   $q1$: We read first matching character and it's a 1
   $q2$: We read any symbol so we have $1x$.
   $q3$: We read any symbol so we have $1xx$.
   $q4$: We read first matching character and it's a 0
   $q5$: We read any symbol so we have $0x$.
   $q6$: We read any symbol so we have $0xx$.
   t: We have found the matching characters at distance 3, any remaining suffix is ok

   (b) All strings over $\Sigma = \{0,1,\ldots,9\}$ that contain *both* 374 and 473 as substrings.

   Solution:



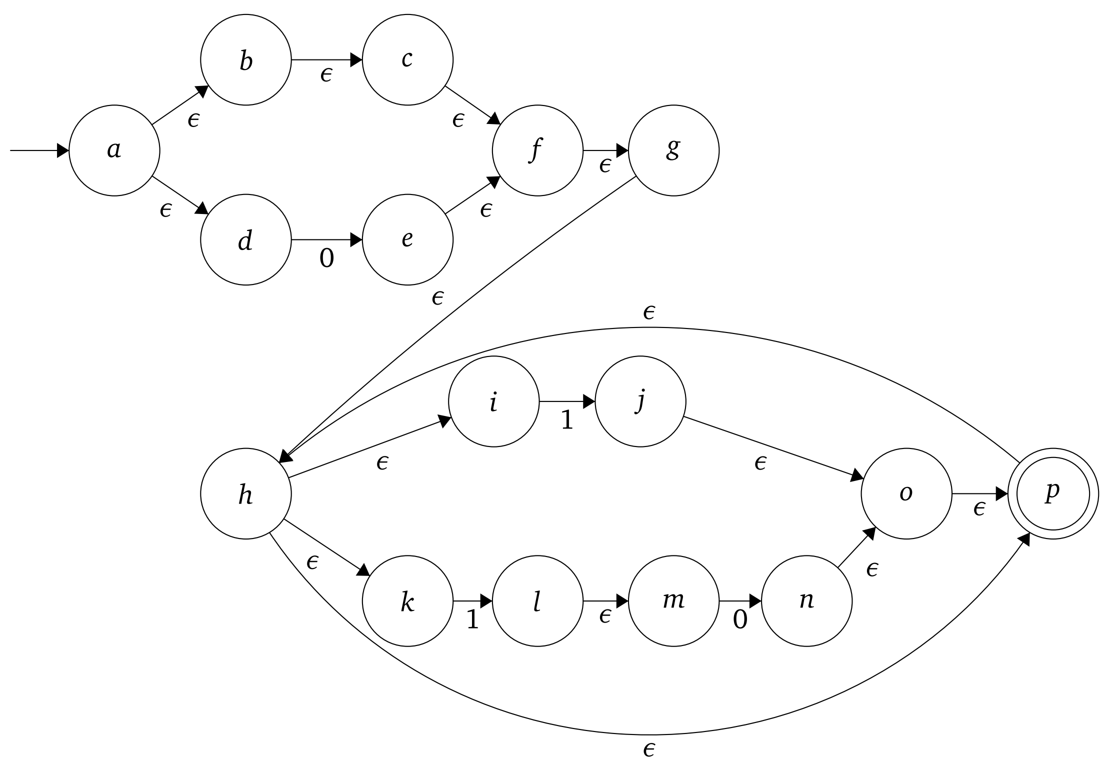
$q0$: Start state.
$q1$: We read first 3 in 374 first.
$q2$: We read 7 for substring 37.
$q3$: We read 4 in a string that has non overlapping 374 - 473.
$q4$: We read a 4, starting the substring 473 after reading 374 (possibly overlap).
$q5$: We read a 7 in the 473 substring after 374.
$q6$: We read first 4 in 473 first.
$q7$: We read 7 for substring 47.
$q8$: We read a 3 in a string that has non overlapping 473 - 374.
$q9$: We read a 3, starting the substring 374 after reading 473 (possibly overlap).
$q10$: We read a 7 in the 374 substring after 473.
t: We read a string that has both substrings 374 and 473.

2. **NFAs to DFAs**. For the following regular expressions, do the following steps:

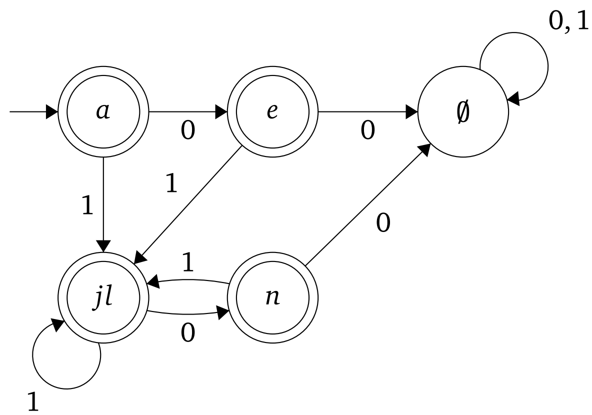
- Construct an NFA corresponding to the regular expression using Thompson's algorithm
- Use the incremental subset construction to convert the NFA to a DFA
- Create another DFA with fewer states to recognize the language

(a) $(\epsilon + 0)(1 + 10)^*$

i. NFA:



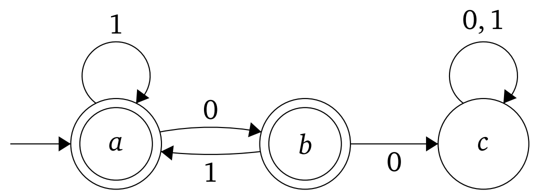
ii. DFA from imcremental subset construction:



q'	$\epsilon-\text{reach}(q')$	$q' \in A$?	$\delta'(q', 0)$	$\delta'(q', 1)$
a	$abcdfghikp$	✓	e	jl
e	$efghikp$	✓	$\emptyset$	jl
jl	$hjlmop$	✓	n	jl
n	$hiknop$	✓	$\emptyset$	jl

Note: the above table was not required but helps understand the solution

iii. DFA:
This language basically means there cannot be two consecutive 0s.



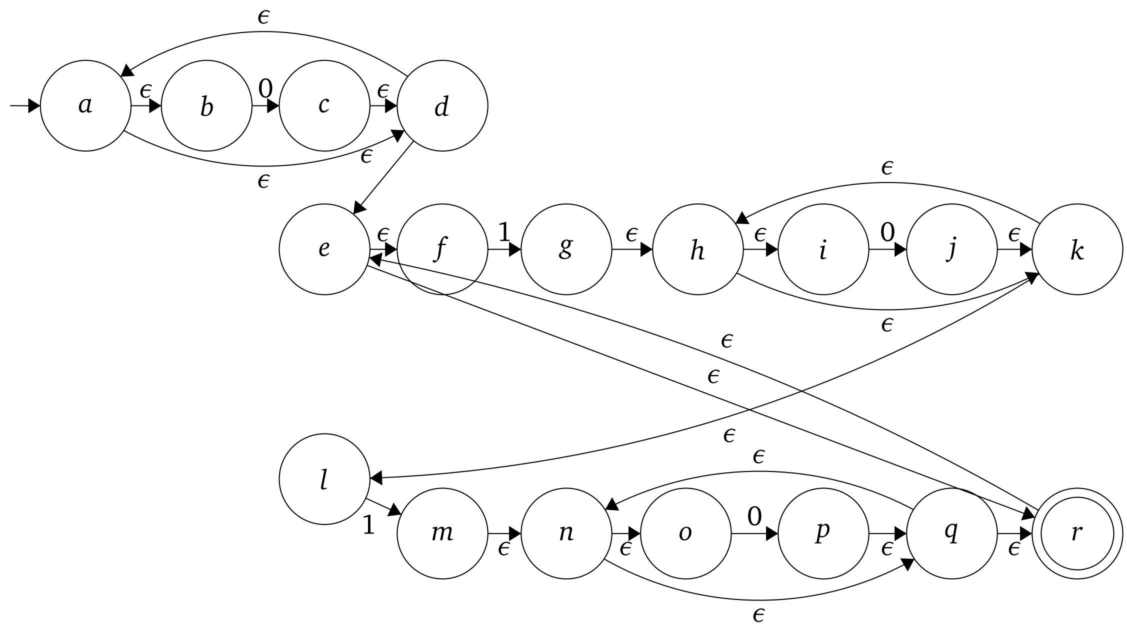
a: No 00 seen and last char was a 1

b: No 00 seen and last char a 0

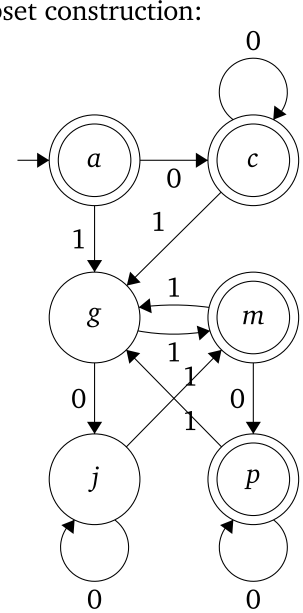
c: 00 has been seen in the input

(b) $0^*(10^*10^*)^*$

i. NFA:

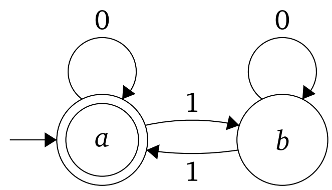


ii. DFA from imcremental subset construction:



q'	$\epsilon-$reach(q')	$q' \in A$?	$\delta'(q', 0)$	$\delta'(q', 1)$
a	$abdefr$	✓	c	g
c	$abcdefr$	✓	c	g
g	$ghikl$		j	m
j	$hijkl$		j	m
m	$efnoqr$	✓	p	g
p	$efnopqr$	✓	p	g

iii. DFA:
The strings in the language should have even number of 1s.



a: Even number of 1's seen
b: Odd number of 1's seen

3. **Palindromes**. In both subproblems below, you need to formally specify the NFA N and formally prove that it accepts the language required by the problem.

   (a) Given a DFA M, define an NFA N such that $L(N) = \{x \in L(M) | x = x^R\}$, i.e., N' accepts the strings in $L(M)$ that are palindromes

   Solution: This is impossible; e.g., if $L(M) = \Sigma^*$, then $L(N)$ is the language of all palindromes, which is not regular and therefore cannot be matched by an NFA. ■

   (b) Given a DFA M, define an NFA N such that $L(N) = \{x \in \Sigma^* | xx^R \in L(M)\}$

   Solution: Let $M = (\Sigma, Q, \delta, s, A)$ be the given DFA. We construct the NFA $N = (\Sigma, Q', \delta', s', A')$ from M as follows.

$$
\begin{aligned}
Q' &= (Q \times Q) \cup \{s'\} \\
s' &\text{ is an explicit state in } Q' \\
A' &= \{(h,h) \mid h \in Q\} \\
\delta'(s', \varepsilon) &= \{(s,a) \mid a \in A\} \\
\delta'((p,q),a) &= \left\{\left(\delta(p,a), q'\right) \;\middle|\; \delta(q',a) = q\right\} \quad \text{for } p,q \in Q, a \in \Sigma
\end{aligned}
$$

   N simultaneously simulates two copies of M on the input string: one that runs normally and one that runs in reverse. To run the normal copy of M on some input symbol, N simply chooses the next state as defined by M's transition function. To run the reverse copy of M on some input symbol, N non-deterministically guesses the previous state from which taking the input symbol transition leads to the current state in the reverse copy.

   - The new start state s' non-deterministically guesses the accepting state $a = \delta^*(s, ww^R)$ without reading any input.
   - State (p,q) means the following:
     - The current state resulting from executing M on the input string w starting from state s is now p.
     - The guess for the current state resulting from executing M in reverse on the input string w starting at some accepting state $a \in A$ is now state q.
   - N accepts w if and only if the input string w leads both the normal and reverse copy of M to some "halfway" state $h \in Q$.

   We can formally prove that N accepts the correct language.

Lemma 1. *For any $n \geq 0$, if $x \in \Sigma^*$ with $|x| = n$, then for any $q_1, q_2 \in Q$:*

$$\delta'^*((q_1,q_2),x) = \{(q_3,q_4) \in Q \times Q | \delta^*(q_1,x) = q_3, \delta^*(q_4,x^R) = q_2\}$$

Proof: Suppose the lemma holds for all $w \in \Sigma^*$ with $|w| < |x|$. If $|x| = 0$ then $x = \epsilon$. $\delta'^*((q_1,q_2),\epsilon) = \{(q_1,q_2)\}$ (note that the only ϵ transition is from s'). Since $\delta^*(q_1,\epsilon) = q_1$ and $\delta^*(q_2,\epsilon^R) = \delta^*(q_2,\epsilon) = q_2$, the lemma holds for this case.

If $|x| > 0$ then $x = aw$ for $a \in \Sigma, w \in \Sigma^*$ with $|w| < |x|$. Then

$$\delta'^*((q_1,q_2),x) = \delta'^*((q_1,q_2),ax) = \bigcup_{(q',q'') \in \delta'((q_1,q_2),a)} \delta'^*((q',q''),w) =$$

$$= \bigcup_{q'' \in Q, \delta(q'',a) = q_2} \delta'^*((\delta(q_1,a),q''),w)$$

By the inductive hypothesis, we know that

$$\delta'^*(\delta(q_1,a),q''),w) = \{(q_3,q_4) \in Q \times Q | \delta^*(\delta(q_1,a),w) = q_3, \delta^*(q_4,w^R) = q''\}$$

But $\delta^*(\delta(q_1,a),w) = \delta^*(q_1,aw) = \delta^*(q_1,x)$. Likewise, since $\delta(q'',a) = q_2$, then

$$\delta^*(q_4,x^R) = \delta^*(q_4,w^Ra) = \delta(\delta^*(q_4,w^R),a) = \delta(q'',a) = q_2$$

This proves the lemma. □

Now suppose that for some $x \in \Sigma^*$, $xx^R \in L(M)$. Then $\delta^*(s,x) = h$ for some $h \in Q$ and $\delta^*(h,x^R) = a$ for some $a \in A$. By the lemma, $(h,h) \in \delta'^*((s,a),x)$. Since $(s,a) \in \epsilon-\mathsf{reach}(s')$, we have $(h,h) \in \delta'^*(s',x)$. And since $(h,h) \in A'$, we have $x \in L(N)$.

Conversely, suppose that $x \in L(N)$. Then for some $h \in Q$, $(h,h) \in \delta'^*(s',x)$. If $x = \epsilon$, then

$$\delta'^*(s',x) = \{s'\} \cup \{(s,a) | a \in A\}$$

Therefore $(h,h) = (s,a)$ for some $a \in A$, which implies that $s \in A$ and so $xx^R = \epsilon\epsilon^R = \epsilon \in L(M)$.

If $x \neq \epsilon$ then $x = cw$ for some $c \in \Sigma$ and $w \in \Sigma^*$. Then:

$$\delta'^*(s',x) = \delta'^*(s',cw) = \bigcup_{p \in \epsilon-\mathsf{reach}(s')} \bigcup r \in \delta'(p,c) \delta^*(r,w)$$

Since s' only has ϵ transitions, there must be some state $(s,a) \in \epsilon-\mathsf{reach}(s')$ such that:

$$(h,h) \in \bigcup r \in \delta'((s,a),c) \delta^*(r,w) = \delta'^*((s,a),cw) = \delta'^*((s,a),x)$$

By the lemma, $\delta^*(s,x) = h$ and $\delta^*(h,x^R) = a$, so $\delta^*(s,xx^R) = a$. Since $a \in A$, we have $xx^R \in L(M)$.

4. **Not to submit:** Recall that for any language L, $\overline{L} = \Sigma^* - L$ is the complement of L. In particular, for any NFA N, $\overline{L(N)}$ is the complement of $L(N)$.

Let $N = (Q, \Sigma, \delta, s, A)$ be an NFA, and define the NFA $N_{\text{comp}} = (Q, \Sigma, \delta, s, Q \setminus A)$. In other words we simply complemented the accepting states of N to obtain N_{comp}. Note that if M is DFA then M_{comp} accepts $\Sigma^* - L(M)$. However things are trickier with NFAs.

(a) Describe a concrete example of a machine N to show that $L(N_{\text{comp}}) \neq \overline{L(N)}$. You need to explain for your machine N what $\overline{L(N)}$ and $L(N_{\text{comp}})$ are.

(b) Define an NFA that accepts $\overline{L(N)} - L(N_{\text{comp}})$, and explain how it works.

(c) Define an NFA that accepts $L(N_{\text{comp}}) - \overline{L(N)}$, and explain how it works.

Hint: For all three parts it is useful to classify strings in Σ^* based on whether N takes them to accepting and non-accepting states from s.

Solved problem

4. Let L be an arbitrary regular language. Prove that the language $\mathit{half}(L) := \{w \mid ww \in L\}$ is also regular.

Solution: Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts L. We define a new NFA $M' = (\Sigma, Q', s', A', \delta')$ with ε-transitions that accepts $\mathit{half}(L)$, as follows:

$$Q' = (Q \times Q \times Q) \cup \{s'\}$$
$$s' \text{ is an explicit state in } Q'$$
$$A' = \{(h, h, q) \mid h \in Q \text{ and } q \in A\}$$
$$\delta'(s', \varepsilon) = \{(s, h, h) \mid h \in Q\}$$
$$\delta'((p, h, q), a) = \big\{\big(\delta(p, a), h, \delta(q, a)\big)\big\}$$

M' reads its input string w and simulates M reading the input string ww. Specifically, M' simultaneously simulates two copies of M, one reading the left half of ww starting at the usual start state s, and the other reading the right half of ww starting at some intermediate state h.

- The new start state s' non-deterministically guesses the "halfway" state $h = \delta^*(s, w)$ without reading any input; this is the only non-determinism in M'.
- State (p, h, q) means the following:
  - The left copy of M (which started at state s) is now in state p.
  - The initial guess for the halfway state is h.
  - The right copy of M (which started at state h) is now in state q.
- M' accepts if and only if the left copy of M ends at state h (so the initial non-deterministic guess $h = \delta^*(s, w)$ was correct) and the right copy of M ends in an accepting state.

■

Rubric: 5 points =
- \+ 1 for a formal, complete, and unambiguous description of a DFA or NFA
  - – No points for the rest of the problem if this is missing.
- \+ 3 for a correct NFA
  - – −1 for a single mistake in the description (for example a typo)
- \+ 1 for a *brief* English justification. We explicitly do *not* want a formal proof of correctness, but we do want one or two sentences explaining how the NFA works.