Lab 10 — Tree-based Dynamic Programming

CS/ECE 374 B

October 30, 2019

For these questions, you are given a tree as a linked data structure. Each tree node has a list of children. You may add one or more fields to the node data structure to help your algorithm.

1. Describe a recursive algorithm to find the maximum independent set of a tree. Your algorithm should take a root node as its input and output the size of the maximum independent set of the tree. (Unlike some past problems, you should not need to add any extra arguments to the recursive function, i.e., your recursive calls should all be of the form MIS(n) for some tree node n).

Solution:

```
def MIS(root):
    """ compute the size of the maximum independent set of
    the tree rooted at 'root' """
    # base case
    if len(root.children) == 0:
        return 1
    # compute MIS that excludes root
    excludes = sum(MIS(child) for child in in root.children)

    # compute MIS that includes root
    includes = 1 + sum(MIS(grandchild) for child in root.children
                       for grandchild in child.children)
    return max(includes, excludes)
```

2. Design and analyze a dynamic programming version of your algorithm. Hint: store the intended result in node.mis

Solution: We perform a pre-order traversal of the tree, computing the node.mis field of each node, which represents the MIS of the subtree rooted at that node. Note that we still use recursion to traverse the tree but the algorithm computes the MIS of each subtree exactly once.

```
def MIS_DP(root):
    """ compute the size of the maximum independent set of
    the tree rooted at 'root' and store it in 'root.mis' """
    # base case
    if len(root.children) == 0:
        return 1
    for child in root.children:
        # compute MIS of the children
```

```
        MIS_DP(child)
    # compute MIS that excludes root
    excludes = sum(child.mis for child in root.children)

    # compute MIS that includes root
    includes = 1 + sum(grandchild.mis for child in root.children
                       for grandchild in child.children)
    root.mis = max(includes, excludes)
```

A sligthly tricky part here is analyzing the performance. Note that MIS_DP gets called excatly once for each tree node. However, the execution of the function is not constant time, since it iterates over all children and grandchildren. In the worst case, where a tree with n nodes has a single root node with n-1 children, we execute n-1 additions in MIS_DP(root).

However, notice that each node participates in at most two sums: one in the MIS_DP call of its parent, and one in the MIS_DP call of its grandparent. (Except for the root and its children). Therefore, the total number of additions is still $\Theta(n)$

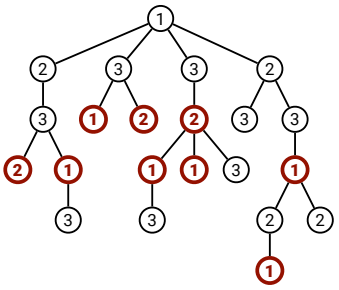
3. Consider now that each node has a weight, node.w. Design an algorithm for finding (the weight of) the maximum *weight* independent set.

Solution: This is essentially the same algorithm, except that we use the weight of the node instead of 1.

```
def MIS_weighted(root):
    """ compute the weight of the maximum-weight independent set of
    the tree rooted at 'root' and store it in 'root.wmis' """
    # base case
    if len(root.children) == 0:
        return root.w
    for child in root.children:
        # compute MIS of the children
        MIS_DP(child)
    # compute MIS that excludes root
    excludes = sum(child.wmis for child in root.children)

    # compute MIS that includes root
    includes = root.w + sum(grandchild.wmis for child in root.children
                            for grandchild in child.children)
    root.wmis = max(includes, excludes)
```

4. Your goal in this problem is to assign each node one of three labels, 1, 2, or 3. Each node *must* have a different label than its parent. The *cost* of an assignment is the number red nodes that have a label smaller than their parent. For example, in the figure below, the red nodes are ones that have a label than their parent, and therefore the cost of this labeling would be 9.



Your goal is to find the (cost of) the minimal cost labeling of a given tree. (The labeling above is *not* the minimal cost one.)

Solution: We define 3 variables: node.cost1, node.cost2, and node.cost3 to represent the minimum cost of a tree rooted at node, where the root is labeled 1, 2, or 3, respectively. We can then recursively compute the costs based on the costs of the children:

```
def min_cost_label(root):
    root.cost1 = 0
    root.cost2 = 0
    root.cost3 = 0
    # base case
    if len(root.children) == 0:
        return 0
    for child in root.children:
        min_cost_label(child)
        root.cost1 += min(child.cost2, child.cost3)
        root.cost2 += min(1+child.cost1, child.cost3)
        root.cost3 += min(1+child.cost1, 1+child.cost2)
    return min(root.cost1, root.cost2, root.cost3)
```