

The following problems ask you to prove some “obvious” claims about recursively-defined string functions. In each case, we want a self-contained, step-by-step induction proof that builds on formal definitions and prior results, *not* on intuition. In particular, your proofs must refer to the formal recursive definitions of string length and string concatenation:

$$|w| = \begin{cases} 0 & \text{if } w = \varepsilon \\ 1 + |x| & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

$$wz = w \bullet z = \begin{cases} z & \text{if } w = \varepsilon \\ a \bullet (x \bullet z) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x. \end{cases}$$

You may freely use the following results, which are proved in the lecture notes:

**Lemma 1:**  $w \bullet \varepsilon = w$  for all strings  $w$ .

**Lemma 2:**  $|w \bullet x| = |w| + |x|$  for all strings  $w$  and  $x$ .

**Lemma 3:**  $(w \bullet x) \bullet y = w \bullet (x \bullet y)$  for all strings  $w$ ,  $x$ , and  $y$ .

The *reversal*  $w^R$  of a string  $w$  is defined recursively as follows:

$$w^R = \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ x^R \bullet a & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x. \end{cases}$$

For example,  $STRESSED^R = DESSERTS$  and  $WTF374^R = 473FTW$ .

- 1** Prove that  $|w| = |w^R|$  for every string  $w$ .
- 2** Prove that  $(w \bullet z)^R = z^R \bullet w^R$  for all strings  $w$  and  $z$ .
- 3** Prove that  $(w^R)^R = w$  for every string  $w$ .

(**Hint:** You need #2 to prove #3, but you may find it easier to solve #3 first.)

**To think about later:** Let  $\#(a, w)$  denote the number of times symbol  $a$  appears in string  $w$ . For example,  $\#(X, WTF374) = 0$  and  $\#(0, 000010101010010100) = 12$ .

- 4** Give a formal recursive definition of  $\#(a, w)$ .
- 5** Prove that  $\#(a, w \bullet z) = \#(a, w) + \#(a, z)$  for all symbols  $a$  and all strings  $w$  and  $z$ .
- 6** Prove that  $\#(a, w^R) = \#(a, w)$  for all symbols  $a$  and all strings  $w$ .