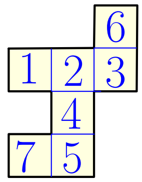
1 Consider the following "maze":



A robot starts at position 1 – where at every point in time it is allowed to move only to adjacent cells. The input is a sequence of commands V (move vertically) or H (move horizontally), where the robot is required to move if it gets such a command. If it is in location 2, and it gets a V command then it must move down to location 4. However, if it gets command H while being in location 2 then it can move either to location 1 or 3, as it chooses.

An input is ***invalid***, if the robot get stuck during the execution of this sequence of commands, for any sequence of choices it makes. For example, starting at position 1, the input HVH is invalid. (The robot was so badly designed, that if it gets stuck, it explodes and no longer exists.)

1.A. Starting at position 1, consider the (command) input HVV. Which location might the robot be in? (Same for $HVVV$ and $HVVVH$.)

1.B. Draw an NFA that accepts all valid inputs.

1.C. The robot *solves* the maze if it arrives (at any point in time) to position 7. Draw an NFA that accepts all inputs that are solutions to the maze.

1.D. (Extra - not for discussion section.) Write a regular expression which is all inputs that are valid solutions to the maze.
(See here for notes of how to solve such a question.)

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2 Let $L = \{w \in \{a, b\}^* \mid a$ appears in some position i of w, and a b appears in position $i + 2\}$.

2.A. Create an NFA N for L with at most four states.

2.B. Using the "power-set" construction, create a DFA M from N. Rather than writing down the sixteen states and trying to fill in the transitions, build the states as needed, because you won't end up with unreachable or otherwise superfluous states.

2.C. Now directly design a DFA M' for L with only five states, and explain the relationship between M and M'.

3 Let L be an arbitrary regular language. Prove that the language $reverse(L) := \left\{w^R \mid w \in L\right\}$ is regular. *Hint:* Consider a DFA M that accepts L and construct a NFA that accepts $reverse(L)$.

4 Let L be an arbitrary regular language. Prove that the language $insert1(L) := \{x1y \mid xy \in L\}$ is regular.
Intuitively, $insert1(L)$ is the set of all strings that can be obtained from strings in L by inserting exactly one 1. For example, if $L = \{\varepsilon, OOK!\}$, then $insert1(L) = \{1, 1OOK!, O1OK!, OO1K!, OOK1!, OOK!1\}$.

<u>Work on these later:</u>

5 Prove that the language $\mathit{delete1}(L) := \{xy \mid x1y \in L\}$ is regular.

Intuitively, $\mathit{delete1}(L)$ is the set of all strings that can be obtained from strings in L by deleting exactly one 1. For example, if $L = \{101101, 00, \varepsilon\}$, then $\mathit{delete1}(L) = \{01101, 10101, 10110\}$.

6 Consider the following recursively defined function on strings:

$$\mathit{stutter}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ aa \bullet \mathit{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x \end{cases}$$

Intuitively, $\mathit{stutter}(w)$ doubles every symbol in w. For example:

- $\mathit{stutter}(PRESTO) = PPRREESSTTOO$
- $\mathit{stutter}(HOCUS\diamond POCUS) = HHOOCCUUSS\diamond\diamond PPOOCCUUSS$

Let L be an arbitrary regular language.

1. Prove that the language $\mathit{stutter}^{-1}(L) := \{w \mid \mathit{stutter}(w) \in L\}$ is regular.
2. Prove that the language $\mathit{stutter}(L) := \{\mathit{stutter}(w) \mid w \in L\}$ is regular.

7 Consider the following recursively defined function on strings:

$$\mathit{evens}(w) := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \varepsilon & \text{if } w = a \text{ for some symbol } a \\ b \cdot \mathit{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x \end{cases}$$

Intuitively, $\mathit{evens}(w)$ skips over every other symbol in w. For example:

- $\mathit{evens}(EXPELLIARMUS) = XELAMS$
- $\mathit{evens}(AVADA\diamond KEDAVRA) = VD\diamond EAR$.

Once again, let L be an arbitrary regular language.

1. Prove that the language $\mathit{evens}^{-1}(L) := \{w \mid \mathit{evens}(w) \in L\}$ is regular.
2. Prove that the language $\mathit{evens}(L) := \{\mathit{evens}(w) \mid w \in L\}$ is regular.