Give regular expressions for each of the following languages over the alphabet $\{0, 1\}$.

1 All strings containing the substring 000.

Solution: $(0 + 1)^*000(0 + 1)^*$

2 All strings *not* containing the substring 000.

Solution: $(1 + 01 + 001)^*(\varepsilon + 0 + 00)$

Solution: $(\varepsilon + 0 + 00)(1(\varepsilon + 0 + 00))^*$

3 All strings in which every run of 0s has length at least 3.

Solution: $(1 + 0000^*)^*$

Solution: $(\varepsilon + 1)((\varepsilon + 0000^*)1)^*(\varepsilon + 0000^*)$

4 All strings in which 1 does not appear after a substring 000.

Solution: $(1 + 01 + 001)^*0^*$

5 All strings containing at least three 0s.

Solution: $(0 + 1)^*0(0 + 1)^*0(0 + 1)^*0(0 + 1)^*$

Solution: $1^*01^*01^*0(0 + 1)^*$ or $(0 + 1)^*01^*01^*01^*$

6 Every string except 000. (**Hint:** Don't try to be clever.)

Solution: Every string $w \neq 000$ satisfies one of three conditions: Either $|w| < 3$, or $|w| = 3$ and $w \neq 000$, or $|w| > 3$. The first two cases include only a finite number of strings, so we just list them explicitly. The last case includes *all* strings of length at least 4.

$$\begin{gathered} \varepsilon + 0 + 1 + 00 + 01 + 10 + 11 \\ + 001 + 010 + 011 + 100 + 101 + 110 + 111 \\ + (1 + 0)(1 + 0)(1 + 0)(1 + 0)(1 + 0)^* \end{gathered}$$

Solution: $\varepsilon + 0 + 00 + \big(1 + 01 + 001 + 000(1 + 0)\big)(1 + 0)^*$

7 All strings w such that *in every prefix of* w, the number of 0s and 1s differ by at most 1.

Solution: Equivalently, strings that alternate between 0s and 1s: $(01 + 10)^*(\varepsilon + 0 + 1)$

8 (**Hard.**) All strings containing at least two 0s and at least one 1.

Solution: There are three possibilities for how such a string can begin:

- Start with 00, then any number of 0s, then 1, then anything.
- Start with 01, then any number of 1s, then 0, then anything.
- Start with 1, then a substring with exactly two 0s, then anything.

All together: $000^*1(0 + 1)^* \ + \ 011^*0(0 + 1)^* \ + \ 11^*01^*0(0 + 1)^*$
Or equivalently: $\big(000^*1 \ + \ 011^*0 \ + \ 11^*01^*0\big)(0 + 1)^*$

Solution:

There are three possibilities for how the three required symbols are ordered:

- Contains a 1 before two 0s: $(0+1)^*\,1\,(0+1)^*\,0\,(0+1)^*\,0\,(0+1)^*$
- Contains a 1 between two 0s: $(0+1)^*\,0\,(0+1)^*\,1\,(0+1)^*\,0\,(0+1)^*$
- Contains a 1 after two 0s: $(0+1)^*\,0\,(0+1)^*\,0\,(0+1)^*\,1\,(0+1)^*$

So putting these cases together, we get the following:

$$\begin{aligned}&(0+1)^*\,1\,(0+1)^*\,0\,(0+1)^*\,0\,(0+1)^*\\+&(0+1)^*\,0\,(0+1)^*\,1\,(0+1)^*\,0\,(0+1)^*\\+&(0+1)^*\,0\,(0+1)^*\,0\,(0+1)^*\,1\,(0+1)^*\end{aligned}$$

9 (**Hard.**) All strings w such that *in every prefix of* w, the number of 0s and 1s differ by at most 2.

Solution: $\big(0(01)^*1 + 1(10)^*0\big)^* \cdot \big(\varepsilon + 0(01)^*(0+\varepsilon) + 1(10)^*(1+\varepsilon)\big)$

10 (**Really hard.**) All strings in which the substring 000 appears an even number of times.
(For example, 0001000 and 0000 are in this language, but 00000 is not.)

Solution:

Every string in $\{0,1\}^*$ alternates between (possibly empty) blocks of 0s and individual 1s; that is, $\{0,1\}^* = (0^*1)^*0^*$. Trivially, every 000 substring is contained in some block of 0s. Our strategy is to consider which blocks of 0s contain an even or odd number of 000 substrings.

We have

Let X denote the set of all strings in 0^* with an even number of 000 substrings. We easily observe that

$$X = \{0^n \mid n = 1 \text{ or } n \text{ is even}\} = 0 + (00)^*.$$

Observe that X contains the empty string.

Let Z_{odd} denote the set of all strings in 0^* with an *odd* number of 000 substrings. We easily observe that

$$Z_{\text{odd}} = \{0^n \mid n > 1 \text{ and } n \text{ is odd}\} = 000(00)^*.$$

We have that $0^* = X + Z_{\text{odd}}$ and therefore $\{0,1\}^* = ((X+Z_{\text{odd}})1)^*(X+Z_{\text{odd}})$.

Finally, let L denote the set of all strings in $\{0,1\}^*$ with an even number of 000 substrings. A string $w \in \{0,1\}^*$ is in $L \iff$ an **even** number of blocks of 0s in w are in Z_{odd}. The remaining blocks of 0s are all in X. To keep things "simpler", let

$$E = (X1)^* = \Big(\big(0+(00)^*\big)1\Big)^*$$

be a run of blocks with even number of 000, ending with a 1 if it is non-empty.

We thus have that

$$\begin{aligned}L = &\, Z_{\text{odd}}1Z_{\text{odd}}\\&+ EZ_{\text{odd}}1EZ_{\text{odd}}+\\&+ EZ_{\text{odd}}1EZ_{\text{odd}}1E\\&+ (EZ_{\text{odd}}1EZ_{\text{odd}}1E)^*\\&+ EZ_{\text{odd}}1(EZ_{\text{odd}}1EZ_{\text{odd}}1E)^*Z_{\text{odd}}\end{aligned}$$

Setting $M = (EZ_{\mathrm{odd}}1EZ_{\mathrm{odd}}1E)^*$, this simplifies to

$$L = M + EZ_{\mathrm{odd}}1MZ_{\mathrm{odd}}.$$

To see why this is correct, consider the last run of zeros with odd number of 000. If it is not the last run of zeros, in the string, then one can argue that M applies. Otherwise, the other expression applies, but then we need to get the first occurrence of Z_{odd} before entering M – thus, the second expression.

Plugging in $E = \Big((0 + (00)^*)1\Big)^*$ and $Z_{\mathrm{odd}} = 000(00)^*$, yields

$$M = \left(\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^*\right)^*$$

As such, L is

$$\begin{aligned} L = &\left(\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^*\right)^* \\ &+ \Big((0 + (00)^*)1\Big)^* 000(00)^* \\ &\qquad 1\left(\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^* 000(00)^*1\Big((0 + (00)^*)1\Big)^*\right)^* 000(00)^*, \end{aligned}$$

which I am sure was your first guess.

Whew!