6.2
When two states are equivalent?

Equivalence between states

Definition

$M = (Q, \Sigma, \delta, s, A)$: DFA.
Two states $p, q \in Q$ *are* equivalent *if for all strings* $w \in \Sigma^*$, *we have that*

$$\delta^*(p, w) \in A \iff \delta^*(q, w) \in A.$$

One can merge any two states that are equivalent into a single state.

Distinguishing between states

Definition

$M = (Q, \Sigma, \delta, s, A)$: DFA.
Two states $p, q \in Q$ *are* distinguishable *if there exists a string* $w \in \Sigma^*$*, such that*

$$\delta^*(p, w) \in A \qquad \text{and} \qquad \delta^*(q, w) \notin A.$$

or

$$\delta^*(p, w) \notin A \qquad \text{and} \qquad \delta^*(q, w) \in A.$$

Distinguishable prefixes

$M = (Q, \Sigma, \delta, s, A)$: DFA
Idea: Every string $w \in \Sigma^*$ defines a state $\nabla w = \delta^*(s, w)$.

Definition

Two strings $u, w \in \Sigma^*$ are distinguishable for M (or $L(M)$) if ∇u and ∇w are distinguishable.

Definition (Direct restatement)

Two prefixes $u, w \in \Sigma^*$ are distinguishable for a language L if there exists a string x, such that $ux \in L$ and $wx \notin L$ (or $ux \notin L$ and $wx \in L$).

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Distinguishable means different states

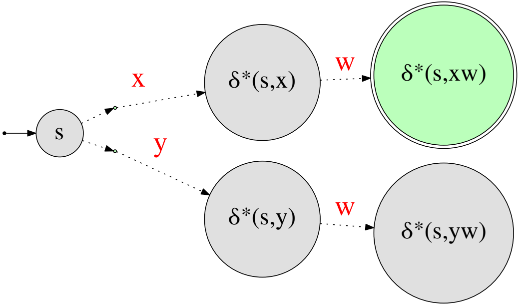
Lemma

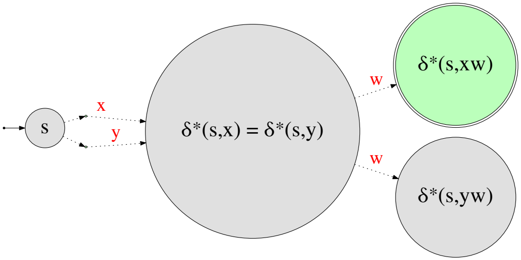
L: regular language.
$M = (Q, \Sigma, \delta, s, A)$: DFA for L.
If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x) \in Q$ and $\nabla y = \delta^*(s, y) \in Q$

Proof by a figure

Possible	Not possible





Distinguishable strings means different states: Proof

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Proof.

Assume for the sake of contradiction that $\nabla x = \nabla y$.
By assumption $\exists w \in \Sigma^*$ such that $\nabla xw \in A$ and $\nabla yw \notin A$.
$\implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w)$
$= \delta^*(s, yw) = \nabla yw \notin A$.
$\implies A \ni \nabla yw \notin A$. Impossible!
Assumption that $\nabla x = \nabla y$ is false. □

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Review questions...

1. Prove for any $i \neq j$ then 0^i and 0^j are distinguishable for the language $\{0^k 1^k \mid k \geq 0\}$.
2. Let L be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for L. Prove that any DFA for L must have at least k states.
3. Prove that $\{0^k 1^k \mid k \geq 0\}$ is not regular.

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THE END

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(for now)