9.3
The halting theorem

Encodings

M: Turing machine
$\langle M \rangle$: a binary string uniquely describing M (i.e., it is a number.
w: An input string.
$\langle M, w \rangle$: A unique binary string encoding both M and input w.

$$\mathrm{A_{TM}} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } \mathrm{TM} \text{ and } M \text{ accepts } w \right\}.$$

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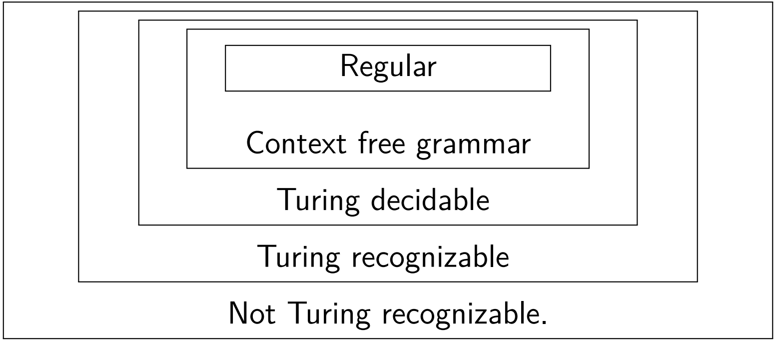
Complexity classes



A_{TM} is TM recognizable...

$$A_{TM} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a TM and } M \text{ accepts } w \right\}.$$

Lemma

A_{TM} is Turing recognizable.

Proof.

Input: $\langle M, w \rangle$.
Using UTM simulate running M on w. If M accepts w then accept, if M rejects then reject. Otherwise, the simulation runs forever. □

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Theorem (The halting theorem.)

$\mathbf{A}_{\mathrm{TM}}$ is not Turing decidable.

Proof: Assume $\mathbf{A}_{\mathrm{TM}}$ is TM decidable...

Halt: TM deciding $\mathbf{A}_{\mathrm{TM}}$. **Halt** always halts, and works as follows:

$$\textbf{Halt}\Big(\langle M, w \rangle\Big) = \begin{cases} \text{accept} & M \text{ accepts } w \\ \text{reject} & M \text{ does not accept } w. \end{cases}$$

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Halting theorem proof continued 1

We build the following new function:

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Flipper(⟨M⟩)
    res ← Halt(⟨M, M⟩)
    if res is accept then
            reject
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Flipper always stops:

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This is absurd. Ridiculous even!
Assumption that **Halt** exists is false. $\implies$ A_{TM} is not TM decidable. □

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But where is the diagonalization argument????

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	...
M_1	**rej**	acc	rej	rej	...
M_2	rej	**acc**	rej	acc	...
M_3	acc	acc	**acc**	rej	...
M_4	rej	acc	acc	**rej**	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

THE END

...

(for now)