11.4.3
Median of Medians

Divide and Conquer Approach

A game of medians

Idea

1. Break input A into many subarrays: $L_1, \ldots L_k$.

2. Find median m_i in each subarray L_i.

3. Find the median x of the medians $m_1, \ldots, m_k$.

4. Intuition: The median x should be close to being a good median of all the numbers in A.

5. Use x as pivot in previous algorithm.

New example

The input:

75	31	13	26	83	110	60	120	63	30	3	41	44	107	30	23	91	17	6	110
68	24	41	26	58	57	61	20	52	45	13	79	86	91	55	66	13	103	36	60
19	40	45	111	56	74	17	95	96	77	29	65	36	96	93	119	9	61	3	9
100	3	88	47	115	107	79	39	109	20	59	25	92	81	36	10	30	113	73	116
72	58	24	16	12	69	40	24	19	92	7	65	75	41	43	117	103	38	8	20

Compute median of the medians (recursive call):

72	74	13	66
31	60	65	30
41	39	75	61
26	63	91	8
58	45	43	60

After partition (pivot **60**):

19	3	13	16	12	57	17	20	19	20	3	25	92	109	96	79	110	69	83	75
41	24	24	26	56	17	40	24	52	30	7	60	77	81	63	61	107	115	111	72
20	31	41	26	58	30	60	39	36	45	13	65	75	91	120	66	74	61	88	68
9	40	45	47	3	13	23	55	30	44	29	65	86	96	95	117	91	103	100	110
36	58	8	6	38	9	10	43	41	36	59	79	92	107	93	119	103	113	73	116

Tail recursive call: Select element of rank **50** out of **56** elements.

19	3	13	16	12	57	17	20	19	20	3	25
41	24	24	26	56	17	40	24	52	30	7	
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Example

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1. Partition array A into $\lceil n/5 \rceil$ lists of 5 items each.
   $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, $\ldots$,
   $L_i = \{A[5i + 1], \ldots, A[5i - 4]\}$, $\ldots$, $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.
2. For each i find median b_i of L_i using brute-force in $O(1)$ time. Total $O(n)$ time
3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
4. Find median b of B

Lemma

Median of B is an approximate median of A. That is, if b is used a pivot to partition A, then $|A_{less}| \leq 7n/10 + 6$ and $|A_{greater}| \leq 7n/10 + 6$.
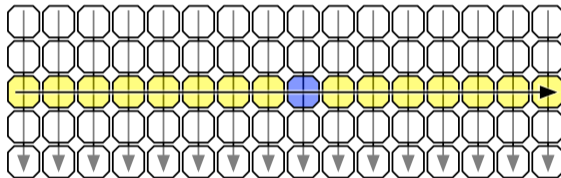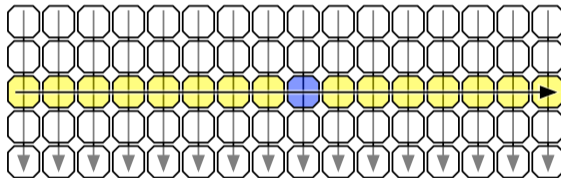
Choosing the pivot
A clash of medians

1. Partition array A into $\lceil n/5 \rceil$ lists of 5 items each.
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```
select(A, j):
    Form lists L_1, L_2, ..., L_{⌈n/5⌉} where L_i = {A[5i − 4], ..., A[5i]}
    Find median b_i of each L_i using brute-force
    Find median b of B = {b_1, b_2, ..., b_{⌈n/5⌉}}
    Partition A into A_less and A_greater using b as pivot
    if (|A_less|) = j return b
    else if (|A_less|) > j)
        return select(A_less, j)
    else
        return select(A_greater, j − |A_less|)
```

How do we find median of B?

Algorithm for Selection
A storm of medians

```
select(A, j):
    Form lists L₁, L₂, ..., L⌈n/5⌉ where Lᵢ = {A[5i − 4], ..., A[5i]}
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How do we find median of B? Recursively!

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    Form lists L_1, L_2, ..., L_⌈n/5⌉ where L_i = {A[5i − 4], ..., A[5i]}
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select(A, j):
    Form lists L₁, L₂, ..., L⌈n/5⌉ where Lᵢ = {A[5i − 4], ..., A[5i]}
    Find median bᵢ of each Lᵢ using brute-force
    B = [b₁, b₂, ..., b⌈n/5⌉]
    b = select(B, ⌈n/10⌉)
    Partition A into A_less and A_greater using b as pivot
    if (|A_less|) = j return b
    else if (|A_less|) > j)
        return select(A_less, j)
    else
        return select(A_greater, j − |A_less|)
```

THE END

...

(for now)