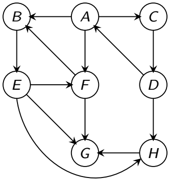
15.4
Directed Graphs and Decomposition

Directed Graphs

Definition

A directed graph $G = (V, E)$ consists of

1. set of vertices/nodes V and
2. a set of edges/arcs $E \subseteq V \times V$.



An edge is an ordered pair of vertices. (u, v) different from (v, u).

Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

1. Road networks with one-way streets.
2. Web-link graph: vertices are web-pages and there is an edge from page p to page p' if p has a link to p'. Web graphs used by Google with PageRank algorithm to rank pages.
3. Dependency graphs in variety of applications: link from x to y if y depends on x. Make files for compiling programs.
4. Program Analysis: functions/procedures are vertices and there is an edge from x to y if x calls y.

Directed Graph Representation

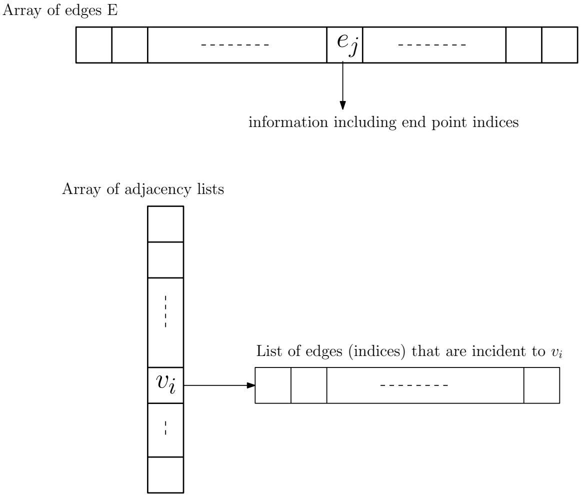
Graph $G = (V, E)$ with n vertices and m edges:

1. **Adjacency Matrix**: $n \times n$ asymmetric matrix A. $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ if $(u, v) \notin E$. $A[u, v]$ is not same as $A[v, u]$.
2. **Adjacency Lists**: for each node u, $Out(u)$ (also referred to as $Adj(u)$) and $In(u)$ store out-going edges and in-coming edges from u.

Default representation is adjacency lists.

A Concrete Representation for Directed Graphs

Concrete representation discussed previously for undirected graphs easily extends to directed graphs.



Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from v_1 to v_k.
   By convention, a single node u is a path of length 0.
2. A **cycle** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$.
   By convention, a single node u is not a cycle.
3. A vertex u can reach v if there is a path from u to v. Alternatively v can be reached from u
4. Let $\text{rch}(u)$ be the set of all vertices reachable from u.

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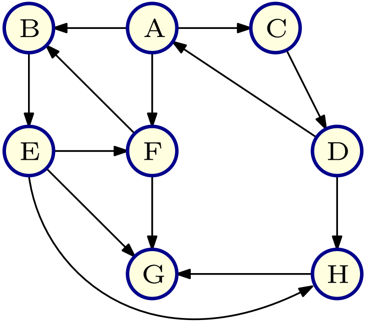
Connectivity contd

Asymmetricity: D can reach B but B cannot reach D



Questions:

1. Is there a notion of connected components?
2. How do we understand connectivity in directed graphs?

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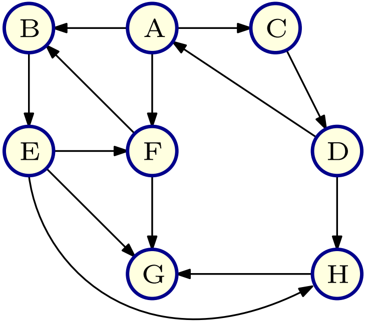

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THE END

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(for now)