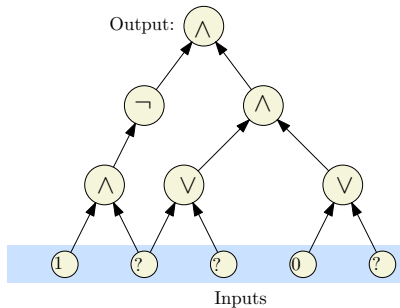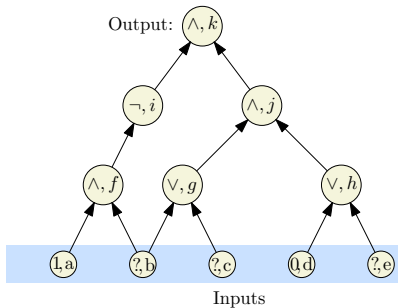# 24.2.3
Reduction from CSAT to SAT

Converting a circuit into a CNF formula
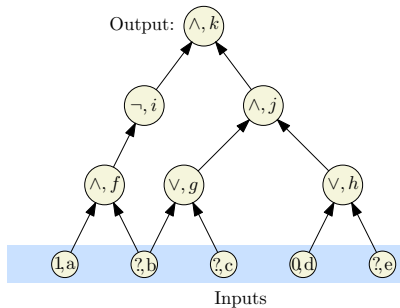
Label the nodes
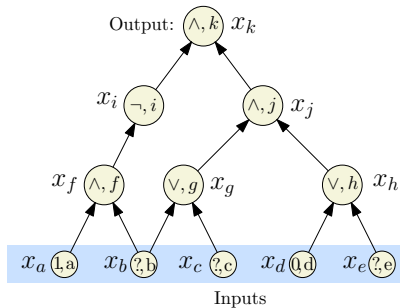

(A) Input circuit

(B) Label the nodes.

Converting a circuit into a CNF formula

Introduce a variable for each node

(B) Label the nodes. (C) Introduce var for each node.

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.

(C) Introduce var for each node.

x_k (Demand a sat' assignment!)

$x_k = x_i \wedge x_j$

$x_j = x_g \wedge x_h$

$x_i = \neg x_f$

$x_h = x_d \vee x_e$

$x_g = x_b \vee x_c$

$x_f = x_a \wedge x_b$

$x_d = 0$

$x_a = 1$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

x_k	x_k
$x_k = x_i \wedge x_j$	$(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$
$x_j = x_g \wedge x_h$	$(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h)$
$x_i = \neg x_f$	$(x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f)$
$x_h = x_d \vee x_e$	$(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)$
$x_g = x_b \vee x_c$	$(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$
$x_f = x_a \wedge x_b$	$(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)$
$x_d = 0$	$\neg x_d$
$x_a = 1$	x_a

From **Lemma 24.8** :

1. $z = \bar{x} \quad \equiv \quad (z \vee x) \wedge (\bar{z} \vee \bar{x})$
2. $(z = x \vee y) \quad \equiv \quad (z \vee \bar{y}) \wedge (z \vee \bar{x}) \wedge (\bar{z} \vee x \vee y)$
3. $(z = x \wedge y) \quad \equiv \quad (z \vee \bar{x} \vee \bar{y}) \wedge (\bar{z} \vee x) \wedge (\bar{z} \vee y)$

Converting a circuit into a CNF formula

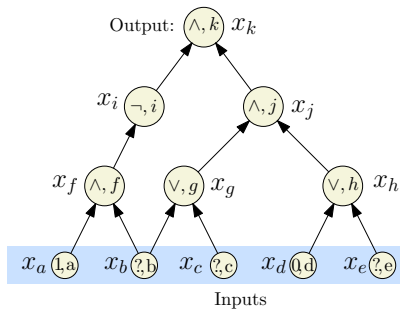Convert each sub-formula to an equivalent CNF formula

x_k	x_k
$x_k = x_i \wedge x_j$	$(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$
$x_j = x_g \wedge x_h$	$(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h)$
$x_i = \neg x_f$	$(x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f)$
$x_h = x_d \vee x_e$	$(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)$
$x_g = x_b \vee x_c$	$(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$
$x_f = x_a \wedge x_b$	$(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)$
$x_d = 0$	$\neg x_d$
$x_a = 1$	x_a

From **Lemma 24.8** :

1. $z = \overline{x} \quad \equiv \quad (z \vee x) \wedge (\overline{z} \vee \overline{x})$
2. $(z = x \vee y) \quad \equiv \quad (z \vee \overline{y}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x \vee y)$
3. $(z = x \wedge y) \quad \equiv \quad (z \vee \overline{x} \vee \overline{y}) \wedge (\overline{z} \vee x) \wedge (\overline{z} \vee y)$

Converting a circuit into a CNF formula

Take the conjunction of all the CNF sub-formulas

$$x_k \wedge (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j)$$
$$\wedge (x_k \vee \neg x_i \vee \neg x_j) \wedge (\neg x_j \vee x_g)$$
$$\wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h)$$
$$\wedge (x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f)$$
$$\wedge (x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e)$$
$$\wedge (\neg x_h \vee x_d \vee x_e) \wedge (x_g \vee \neg x_b)$$
$$\wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$$
$$\wedge (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b)$$
$$\wedge (x_f \vee \neg x_a \vee \neg x_b) \wedge (\neg x_d) \wedge x_a$$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.

Correctness of Reduction

Need to show circuit C is satisfiable if and only if φ_C is satisfiable

$\Rightarrow$ Consider a satisfying assignment a for C
   1. Find values of all gates in C under a
   2. Give value of gate v to variable x_v; call this assignment a'
   3. a' satisfies φ_C (exercise)

$\Leftarrow$ Consider a satisfying assignment a for φ_C
   1. Let a' be the restriction of a to only the input variables
   2. Value of gate v under a' is the same as value of x_v in a
   3. Thus, a' satisfies C

The result

Lemma 24.9.
$CSAT \leq_P SAT \leq_P 3SAT$.

Theorem 24.10.
$CSAT$ is **NP-Complete**.

The result

Lemma 24.9.
CSAT $\leq_P$ *SAT* $\leq_P$ *3SAT*.

Theorem 24.10.
CSAT is **NP-Complete**.

THE END

...

(for now)