Proving Non-regularity

Lecture 6

Thursday, September 10, 2020

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6.1
Not all languages are regular

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? No.

- Each DFA M can be represented as a string over a finite alphabet Σ by appropriate encoding
- Hence number of regular languages is countably infinite
- Number of languages is uncountably infinite
- Hence there must be a non-regular language!

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A direct proof

$L = \{0^i 1^i \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots,\}$

Theorem

L is not regular.

A Simple and Canonical Non-regular Language

$L = \{0^k 1^k \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots, \}$

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Intuition: Any program to recognize L seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?

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Proof by Contradiction

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

What states does M reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \le i < j \le n$.
That is, M is in the same state after reading 0^i and 0^j where $i \neq j$.

M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.
This contradicts the fact that M accepts L. Thus, there is no DFA for L.

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6.2
When two states are equivalent?

Equivalence between states

Definition

$M = (Q, \Sigma, \delta, s, A)$: DFA.
Two states $p, q \in Q$ are equivalent *if for all strings $w \in \Sigma^*$, we have that*

$$\delta^*(p, w) \in A \iff \delta^*(q, w) \in A.$$

One can merge any two states that are equivalent into a single state.

Distinguishing between states

Definition

$M = (Q, \Sigma, \delta, s, A)$: DFA.

Two states $p, q \in Q$ *are* distinguishable *if there exists a string* $w \in \Sigma^*$*, such that*

$$\delta^*(p, w) \in A \qquad \text{and} \qquad \delta^*(q, w) \notin A.$$

or

$$\delta^*(p, w) \notin A \qquad \text{and} \qquad \delta^*(q, w) \in A.$$

Distinguishable prefixes

$M = (Q, \Sigma, \delta, s, A)$: DFA

Idea: Every string $w \in \Sigma^*$ defines a state $\nabla w = \delta^*(s, w)$.

Definition

Two strings $u, w \in \Sigma^*$ are distinguishable for M (or $L(M)$) if ∇u and ∇w are distinguishable.

Definition (Direct restatement)

Two prefixes $u, w \in \Sigma^*$ are distinguishable for a language L if there exists a string x, such that $ux \in L$ and $wx \notin L$ (or $ux \notin L$ and $wx \in L$).

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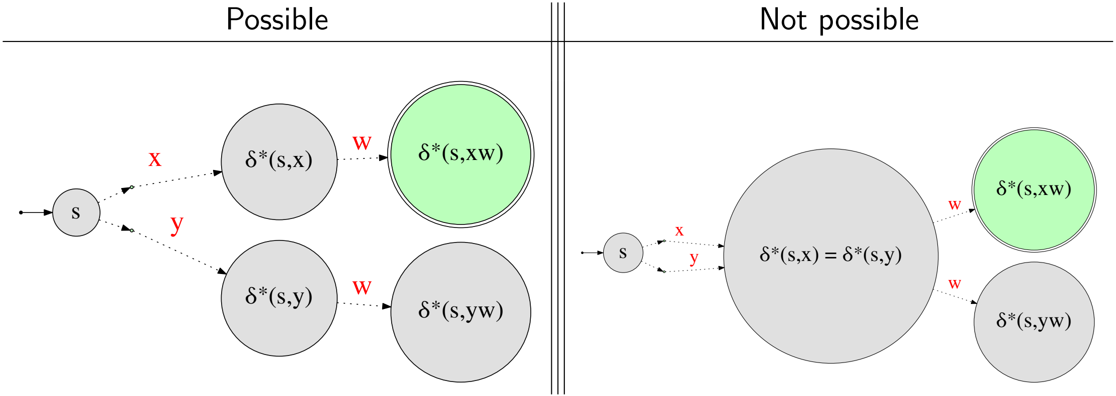
Distinguishable means different states

Lemma

L: regular language.
$M = (Q, \Sigma, \delta, s, A)$: DFA for L.
If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x) \in Q$ and $\nabla y = \delta^*(s, y) \in Q$

Proof by a figure



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Assumption that $\nabla x = \nabla y$ is false. □

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Review questions...

1. Prove for any $i \neq j$ then 0^i and 0^j are distinguishable for the language $\{0^k 1^k \mid k \geq 0\}$.
2. Let L be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for L. Prove that any DFA for L must have at least k states.
3. Prove that $\{0^k 1^k \mid k \geq 0\}$ is not regular.

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6.2.1
Old version: Proving non-regularity

Show non-regularity

Proof structure for showing a language L is not regular:

1. For sake of contradiction, assume it is regular.
2. There exists a finite DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts the language.
3. Showing that there are prefix strings $w_1, w_2, \ldots$ that are all distinguishable.
4. Define $q_i = \nabla w_i = \delta^*(s, w_i)$, for $i = 1, \ldots, \infty$.
5. $\forall i, j : i \neq j$: Since w_i and w_j are distinguishable $\implies q_i \neq q_j$.
6. M has infinite number of states. Impossible!
7. Contradiction to L being regular.

How to prove non-regularity?

Claim: Language L is not regular.

Idea: Show # states in any DFA M for language L has infinite number of states.

Lemma

Consider three strings $x, y, w \in \Sigma^*$.
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If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

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Generalizing the argument

Definition

For a language L over Σ and two strings $x, y \in \Sigma^*$, x and y are distinguishable with respect to L if there is a string $w \in \Sigma^*$ such that exactly one of xw, yw is in L.
x, y are indistinguishable with respect to L if there is no such w.

Example: If $i \neq j$, 0^i and 0^j are distinguishable with respect to $L = \{0^k 1^k \mid k \geq 0\}$

Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w \text{ has } 00 \text{ as a substring}\}$.

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Example: If $i \neq j$, 0^i and 0^j are distinguishable with respect to $L = \{0^k 1^k \mid k \geq 0\}$

Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w \text{ has 00 as a substring}\}$.

Wee Lemma

Lemma

Suppose $L = L(M)$ *for some* DFA $M = (Q, \Sigma, \delta, s, A)$ *and suppose* x, y *are distinguishable with respect to* L. *Then* $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.

Since x, y are distinguishable let w be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then M will either accept both the strings xw, yw, or reject both. But exactly one of them is in L, a contradiction. □

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THE END

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(for now)

6.3
Fooling sets: Proving non-regularity

Fooling Sets

Definition

For a language L over Σ a set of strings F (could be infinite) is a fooling set or distinguishing set for L if every two distinct strings $x, y \in F$ are distinguishable.

Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^k 1^k \mid k \geq 0\}$.

Theorem

Suppose F is a fooling set for L. If F is finite then there is no DFA M that accepts L with less than $|F|$ states.

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Theorem

Suppose F is a fooling set for L. If F is finite then there is no DFA M that accepts L with less than $|F|$ states.

Recall

Already proved the following lemma:

Lemma

L: regular language.
$M = (Q, \Sigma, \delta, s, A)$: DFA for L.
If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x)$.

Proof of theorem

Theorem (Reworded.)

L: A language
F: a fooling set for L.
If F is finite then any DFA M that accepts L has at least $|F|$ states.

Proof.

Let $F = \{w_1, w_2, \ldots, w_m)$ be the fooling set.
Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts L.
Let $q_i = \nabla w_i = \delta^*(s, x_i)$.
By lemma $q_i \neq q_j$ for all $i \neq j$.
As such, $|Q| \geq |\{q_1, \ldots, q_m\}| = |\{w_1, \ldots, w_m\}| = |F|$. □

Proof of theorem

Theorem (Reworded.)

L: A language
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Infinite Fooling Sets

Corollary

If L has an infinite fooling set F then L is not regular.

Proof.

Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists$ M a DFA for L.

Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, # states of $M \geq |F_i| = i$, for all i.

As such, number of states in M is infinite.

Contradiction: DFA = deterministic finite automata. But M not finite. □

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Examples

- $\{0^k 1^k \mid k \geq 0\}$
- {bitstrings with equal number of 0s and 1s}
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- $\{$bitstrings with equal number of 0s and 1s$\}$
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Harder example: The language of squares is not regular

$\{0^{k^2} \mid k \geq 0\}$

Really hard: Primes are not regular

An exercise left for your enjoyment

$\{0^k \mid k \text{ is a prime number}\}$

Hints:

1. Probably easier to prove directly on the automata.
2. There are infinite number of prime numbers.
3. For every $n > 0$, observe that $n!, n! + 1, \ldots, n! + n$ are all composite – there are arbitrarily big gaps between prime numbers.

THE END

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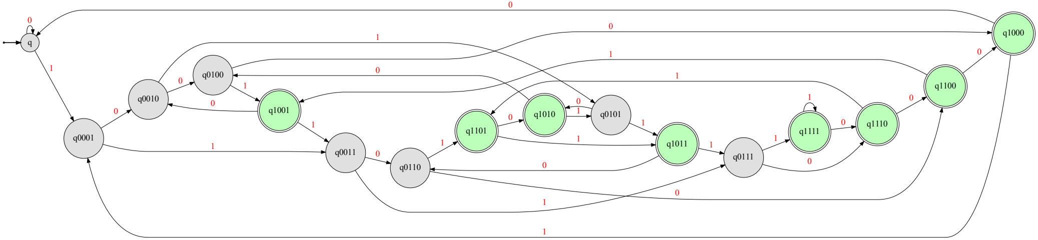
(for now)

6.3.1
Exponential gap in number of states between DFA and NFA sizes

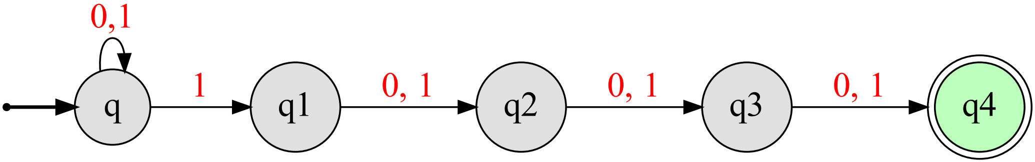
Exponential gap between NFA and DFA size

$L_4 = \{w \in \{0,1\}^* \mid w \text{ has a } 1 \text{ located } 4 \text{ positions from the end}\}$

DFA:



NFA:



Exponential gap between NFA and DFA size

$L_k = \{w \in \{0,1\}^* \mid w \text{ has a 1 } k \text{ positions from the end}\}$

Recall that L_k is accepted by a NFA N with $k+1$ states.

Theorem

Every DFA that accepts L_k has at least 2^k states.

Claim

$F = \{w \in \{0,1\}^* : |w| = k\}$ is a fooling set of size 2^k for L_k.

Why?

- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings

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Every DFA that accepts L_k has at least 2^k states.

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$F = \{w \in \{0,1\}^* : |w| = k\}$ *is a fooling set of size 2^k for L_k.*

Why?

- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings

How do pick a fooling set

How do we pick a fooling set F?

- If x, y are in F and $x \neq y$ they should be distinguishable! Of course.
- All strings in F except maybe one should be prefixes of strings in the language L. For example if $L = \{0^k 1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?

THE END

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(for now)

6.4
Closure properties: Proving non-regularity

Non-regularity via closure properties

$H = \{\text{bitstrings with equal number of 0s and 1s}\}$

$H' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

$H' = H \cap L(0^*1^*)$

Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

Suppose H is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, H' also would be regular. But we know H' is not regular, a contradiction.

Non-regularity via closure properties

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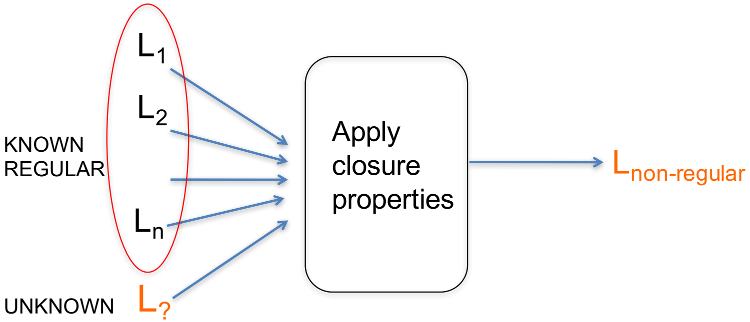
Non-regularity via closure properties

General recipe:



Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

THE END

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(for now)

6.5
Myhill-Nerode Theorem

One automata to rule them all

"Myhill-Nerode Theorem": A regular language L has a unique (up to naming) minimal automata, and it can be computed efficiently once any DFA is given for L.

6.5.1
Myhill-Nerode Theorem: Equivalence between strings

Indistinguishability

Recall:

Definition

For a language L over Σ and two strings $x, y \in \Sigma^*$ we say that x and y are distinguishable with respect to L if there is a string $w \in \Sigma^*$ such that exactly one of xw, yw is in L. x, y are indistinguishable with respect to L if there is no such w.

Given language L over Σ define a relation $\equiv_L$ over strings in Σ^* as follows: $x \equiv_L y$ iff x and y are indistinguishable with respect to L.

Definition

$x \equiv_L y$ *means that* $\forall w \in \Sigma^*: xw \in L \iff yw \in L$.

In words: x is equivalent to y under L.

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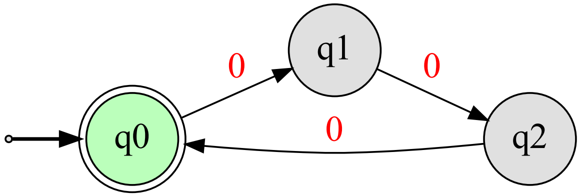
Example: Equivalence classes



Indistinguishability

Claim

$\equiv_L$ *is an equivalence relation over* Σ^*.

Proof.

1. Reflexive: $\forall x \in \Sigma^*: \forall w \in \Sigma^*: xw \in L \iff xw \in L.$ $\implies x \equiv_L x$.
2. Symmetry: $x \equiv_L y$ then $\forall w \in \Sigma^*: xw \in L \iff yw \in L$
   $\forall w \in \Sigma^*: yw \in L \iff xw \in L \implies y \equiv_L x$.
3. Transitivity: $x \equiv_L y$ and $y \equiv_L z$
   $\forall w \in \Sigma^*: xw \in L \iff yw \in L$ and $\forall w \in \Sigma^*: yw \in L \iff zw \in L$
   $\implies \forall w \in \Sigma^*: xw \in L \iff zw \in L$
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□

Equivalences over automatas...

Claim (Just proved.)

$\equiv_L$ *is an equivalence relation over* Σ^*.

Therefore, $\equiv_L$ partitions Σ^* into a collection of equivalence classes.

Definition

L: *A language For a string* $x \in \Sigma^*$, *let*

$$[x] = [x]_L = \{y \in \Sigma^* \mid x \equiv_L y\}$$

be the equivalence class *of* x *according to* L.

Definition

$[L] = \{[x]_L \mid x \in \Sigma^*\}$ *is the set of* equivalence classes *of* L.

Strings in the same equivalence class are indistinguishable

Lemma

Let x, y be two distinct strings.
$x \equiv_L y \iff x, y$ *are indistinguishable for* L.

Proof.

$x \equiv_L y \implies \forall w \in \Sigma^*: xw \in L \iff yw \in L$
x and y are indistinguishable for L.

$x \not\equiv_L y \implies \exists w \in \Sigma^*: xw \in L$ and $yw \notin L$
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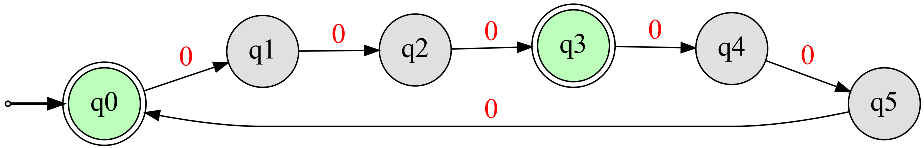
□

All strings arriving at a state are in the same class

Lemma

$M = (Q, \Sigma, \delta, s, A)$ a DFA for a language L.
For any $q \in A$, let $L_q = \{w \in \Sigma^* \mid \nabla w = \delta^*(s, w) = q\}$.
Then, there exists a string x, such that $L_q \subseteq [x]_L$.

An inefficient automata



THE END

...

(for now)

6.5.2
Stating and proving the Myhill-Nerode Theorem

Equivalences over automatas...

Claim (Just proved)

Let x, y be two distinct strings.
$x \equiv_L y \iff x, y$ *are indistinguishable for* L.

Corollary

If $\equiv_L$ *is finite with* n *equivalence classes then there is a fooling set* F *of size* n *for* L.

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If $\equiv_L$ *has infinite number of equivalence classes* $\implies \exists$ *infinite fooling set for* L.
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Equivalence classes as automata

Lemma

For all $x, y \in \Sigma^*$, *if* $[x]_L = [y]_L$, *then for any* $a \in \Sigma$, *we have* $[xa]_L = [ya]_L$.

Proof.

$[x] = [y] \implies \forall w \in \Sigma^*: \; xw \in L \iff yw \in L$

$\implies \forall w' \in \Sigma^*: \; xaw' \in L \iff yaw' \in L \qquad // \; w = aw'$

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Set of equivalence classes

Lemma

If L has n distinct equivalence classes, then there is a DFA that accepts it using n states.

Proof.

Set of states: $Q = [L]$

Start state: $s = [\varepsilon]_L$.

Accept states: $A = \{[x]_L \mid x \in L\}$.

Transition function: $\delta([x]_L, a) = [xa]_L$.

$M = (Q, \Sigma, \delta, s, A)$: The resulting DFA.

Clearly, M is a DFA with n states, and it accepts L. □

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Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

L is regular $\iff$ $\equiv_L$ has a finite number of equivalence classes.
If $\equiv_L$ is finite with n equivalence classes then there is a DFA M accepting L with exactly n states and this is the minimum possible.

Corollary

A language L is non-regular if and only if there is an infinite fooling set F for L.

Algorithmic implication: For every DFA M one can find in polynomial time a DFA M' such that $L(M) = L(M')$ and M' has the fewest possible states among all such DFAs.

What was that all about

Summary: A regular language L has a unique (up to naming) minimal automata, and it can be computed efficiently once any DFA is given for L.

Exercise

1. Given two DFAs M_1, M_2 describe an efficient algorithm to decide if $L(M_1) = L(M_2)$.
2. Given DFA M, and two states q, q' of M, show an efficient algorithm to decide if q and q' are distinguishable. (Hint: Use the first part.)
3. Given a DFA M for a language L, describe an efficient algorithm for computing the minimal automata (as far as the number of states) that accepts L.