Algorithms & Models of Computation

CS/ECE 374, Fall 2020

Greedy Algorithms

Lecture 19

Tuesday, November 3, 2020

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19.1
Greedy algorithms by example

Greedy algorithms

Why don't you do right?

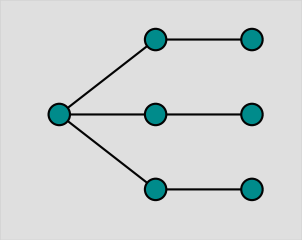
1. **greedy algorithms**: do locally the right thing...
2. ...and they suck.

   Problem: VertexCoverMin

   Instance: Vertex Cover!Minimization
   Question: A graph G.

   Return the smallest subset $S \subseteq V(G)$, s.t. S touches all the edges of G.

3. **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.



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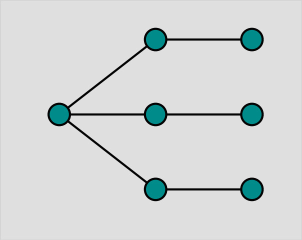

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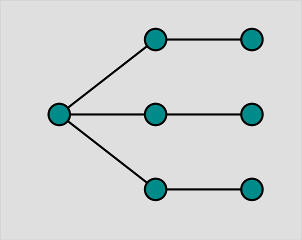

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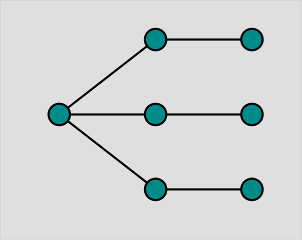
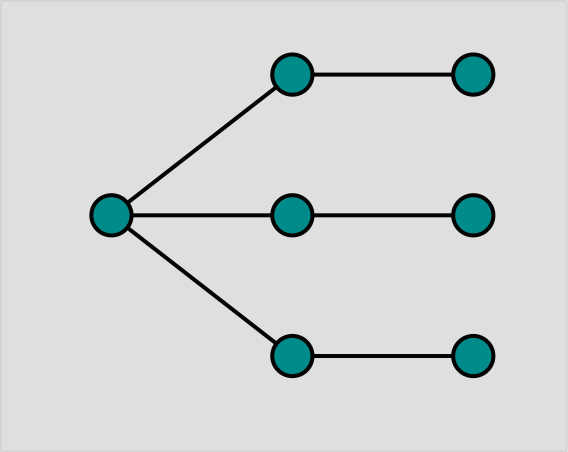


Greedy algorithms

GreedyVertexCover in action...



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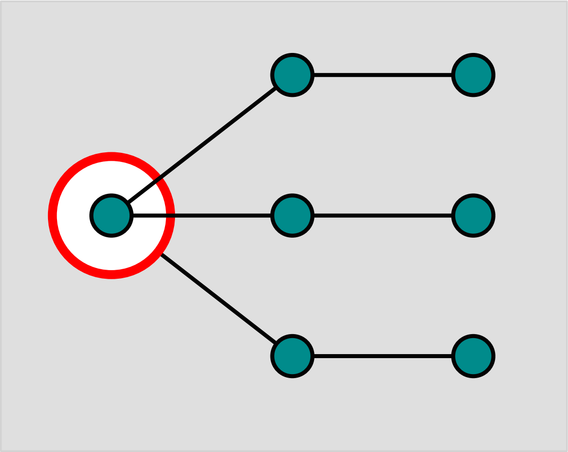

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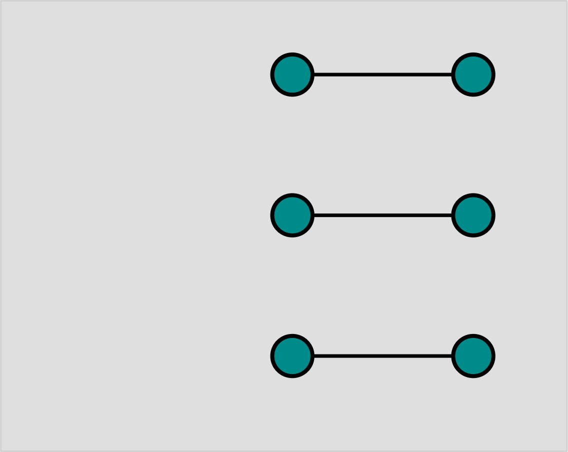

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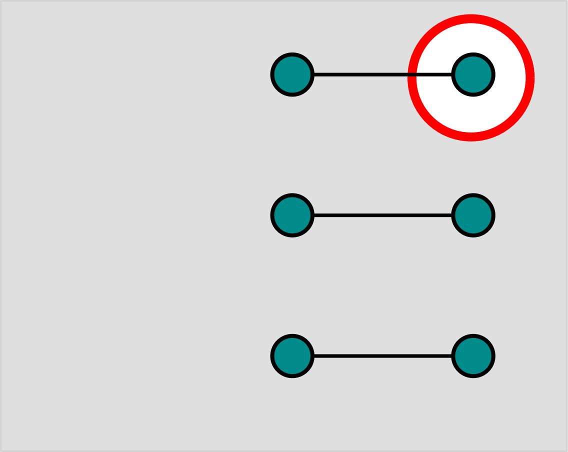

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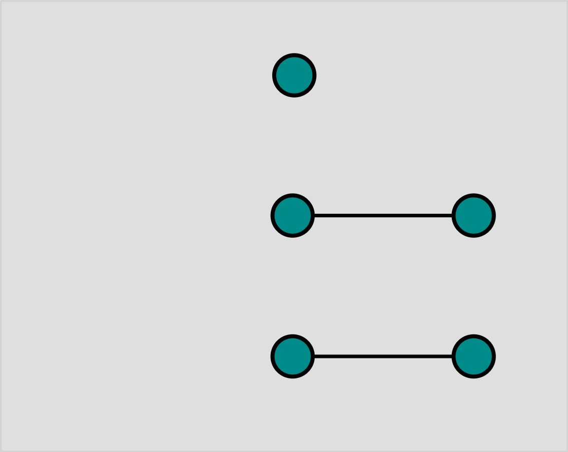

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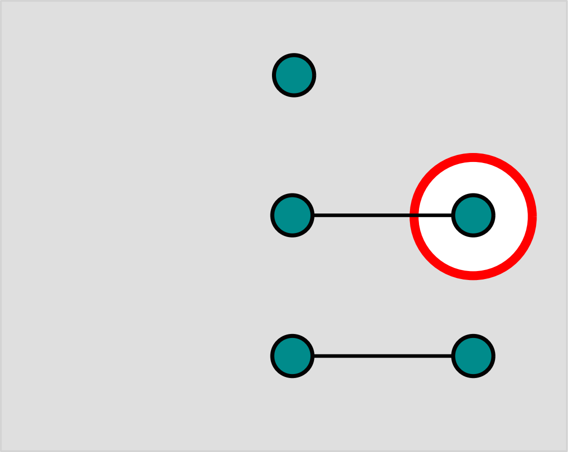

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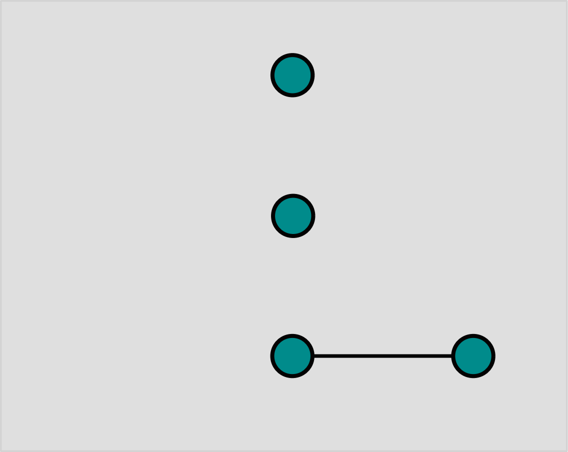

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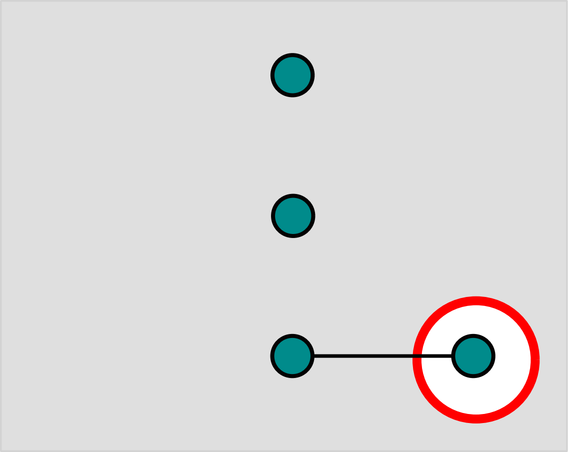

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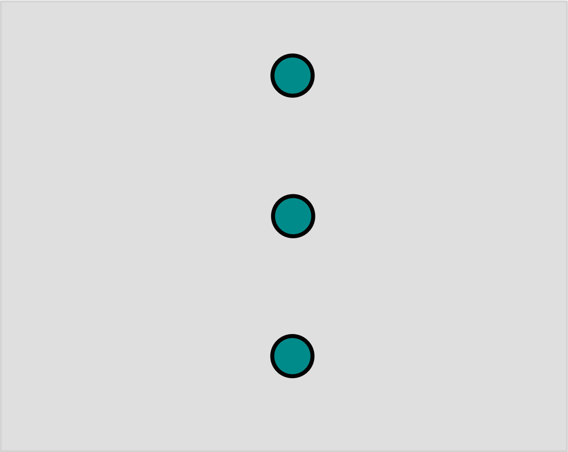

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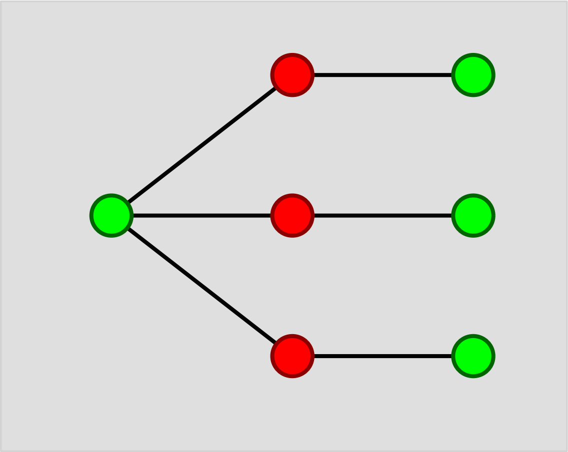

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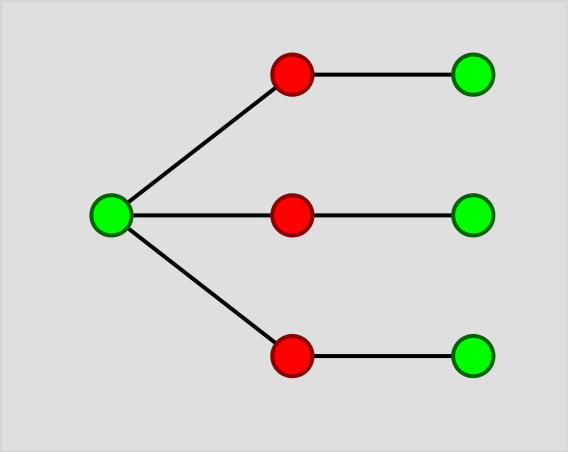

Observation 19.1.

GreedyVertexCover *returns 4 vertices, but opt is 3 vertices.*

Back to GreedyVertexCover

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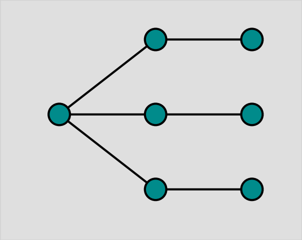
2. Returns 4, but opt is 3!



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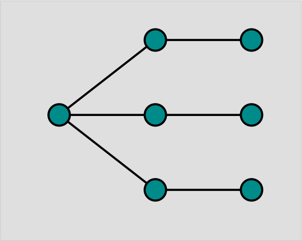

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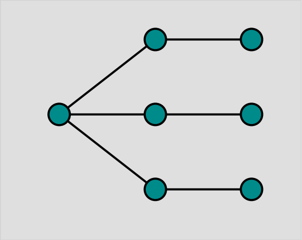

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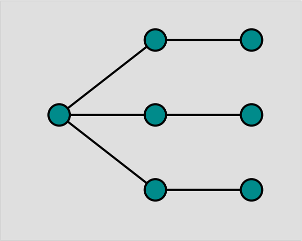

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Greedy Vertex Cover

Theorem 19.2.

There is a graph over n vertices, such that the smallest Vertex Cover *has k vertices, but the greedy algorithm outputs a vertex cover of size $\Theta(k \log n)$ approximation.*

Proof: Outside the scope of this class...
...left as a **hard** exercise to the interested reader.

Vertex Cover is **NP-Hard**: Believe it requires exponential time to solve exactly.

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19.2
Greedy Algorithms: Tools and Techniques

What is a Greedy Algorithm?

No real consensus on a universal definition.

Greedy algorithms:

1. make decision incrementally in small steps without backtracking
2. decision at each step is based on improving local or current state in a myopic fashion without paying attention to the global situation
3. decisions often based on some fixed and simple priority rules

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Pros and Cons of Greedy Algorithms

Pros:

1. Usually (too) easy to design greedy algorithms
2. Easy to implement and often run fast since they are simple
3. Several important cases where they are effective/optimal
4. Lead to a first-cut heuristic when problem not well understood

Cons:

1. **Very often** greedy algorithms don't work. Easy to lull oneself into believing they work
2. Many greedy algorithms possible for a problem and no structured way to find effective ones

CS 374: Every greedy algorithm needs a proof of correctness

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Greedy Algorithm Types

Crude classification:

1. Non-adaptive: fix some ordering of decisions a priori and stick with the order
2. Adaptive: make decisions adaptively but greedily/locally at each step

Plan:

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19.3
Scheduling Jobs to Minimize Average Waiting Time

The Problem

- n jobs $J_1, J_2, \ldots, J_n$.
- Each J_i has non-negative processing time p_i
- One server/machine/person available to process jobs.
- Schedule/order jobs to min. total or average waiting time
- Waiting time of J_i in schedule σ: sum of processing times of all jobs scheduled before J_i

	J_1	J_2	J_3	J_4	J_5	J_6
time	3	4	1	8	2	6

Example: schedule is $J_1, J_2, J_3, J_4, J_5, J_6$. Total waiting time is

$$0 + 3 + (3 + 4) + (3 + 4 + 1) + (3 + 4 + 1 + 8) + \ldots =$$

Optimal schedule: Shortest Job First. $J_3, J_5, J_1, J_2, J_6, J_4$.

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Optimality of Shortest Job First (SJF)

Theorem 19.1.

Shortest Job First gives an optimum schedule for the problem of minimizing total waiting time.

Proof strategy: exchange argument

Assume without loss of generality that job sorted in increasing order of processing time and hence $p_1 \leq p_2 \leq \ldots \leq p_n$ and SJF order is $J_1, J_2, \ldots, J_n$.

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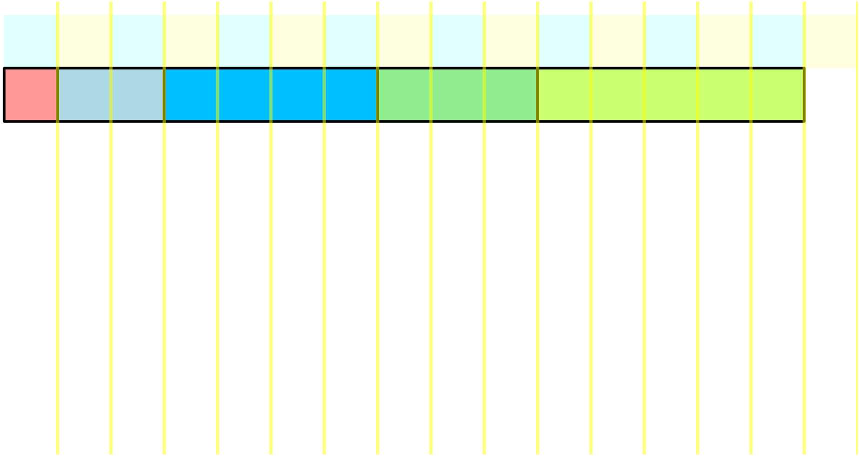

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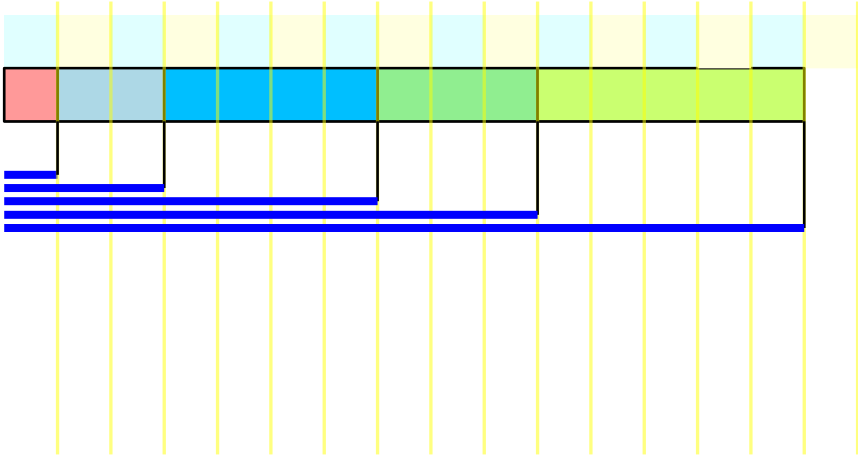

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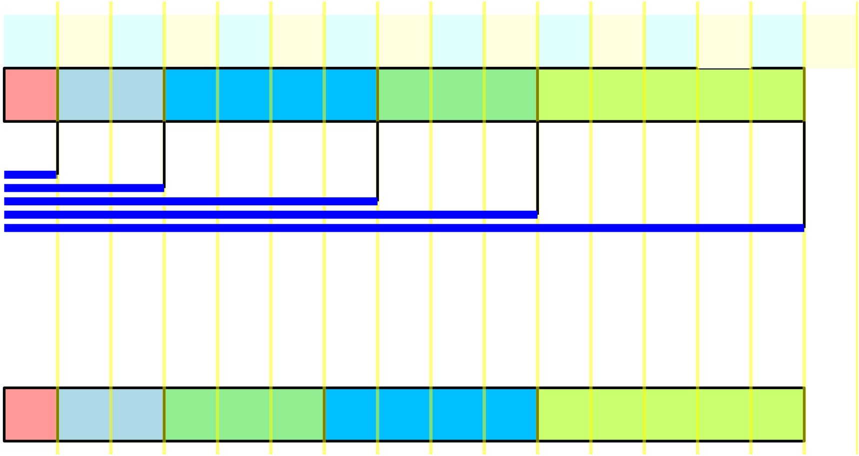

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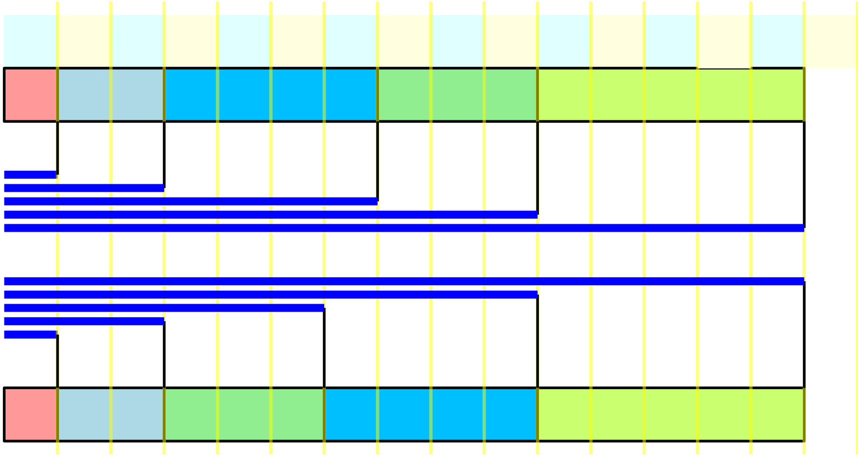

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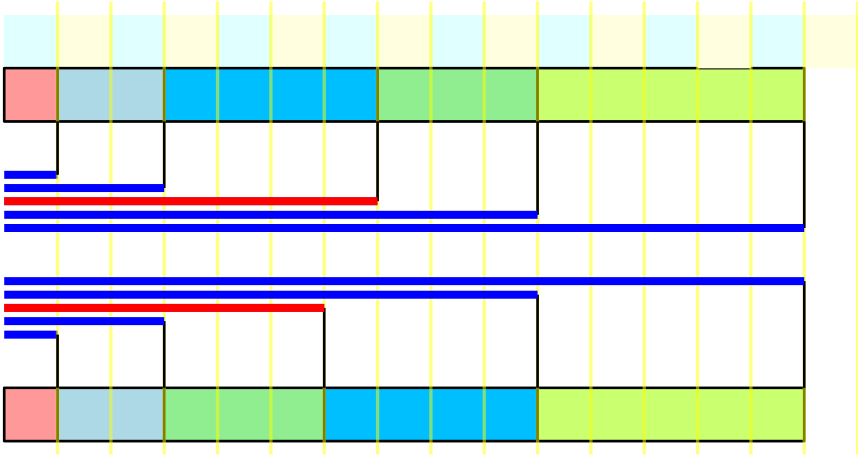

Inversions

Definition 19.2.

A schedule $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ has an inversion if there are jobs J_a and J_b such that S schedules J_a before J_b, but $p_a > p_b$.

Claim 19.3.

If a schedule has an inversion then there is an inversion between two adjacent scheduled jobs.

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Proof of optimality of SJF

SJF = Shortest Job First

Recall SJF order is $J_1, J_2, \ldots, J_n$.

- Let $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ be an optimum schedule with fewest inversions.
- If schedule has no inversions then it is identical to SJF schedule and we are done.
- Otherwise there is an $1 \leq \ell < n$ such that $i_\ell > i_{\ell+1}$ since schedule has inversion among two adjacent scheduled jobs

Claim 19.4.

The schedule obtained from $J_{i_1}, J_{i_2}, \ldots, J_{i_n}$ by exchanging/swapping positions of jobs J_{i_ℓ} and $J_{i_{\ell+1}}$ is also optimal and has one fewer inversion.

Assuming claim we obtain a contradiction and hence optimum schedule with fewest inversions must be the SJF schedule.

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Exercise: A Weighted Version

- n jobs $J_1, J_2, \ldots, J_n$. J_i has non-negative processing time p_i and a non-negative weight w_i
- One server/machine/person available to process jobs.
- Schedule/order the jobs to minimize total or average waiting time
- Waiting time of J_i in schedule σ: sum of processing times of all jobs scheduled before J_i
- Goal: minimize total weighted waiting time.
- Formally, compute a permutation π that minimizes $\sum_{i=1}^{n}\left(\sum_{j=1}^{i-1} p_{\pi(j)}\right) w_{\pi(i)}$.

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19.3.1

Exercise: Scheduling Jobs to Minimize Weighted Average Waiting Time

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	Job 1 first	Job 2 first

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$\omega_i = w_i / p_i$: Price per processing unit in dollars

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$\omega_i = w_i / p_i$: Price per processing unit in dollars

Exercise: A Weighted Version

Consider two jobs p_1, p_2 of weight w_1 and w_2. We have two possibilities:

	Job 1 first	Job 2 first
Pricing	$0 \cdot w_1 + p_1 w_2$	$0 w_2 + p_2 w_1$
Equivalent to	$p_1 w_2$	$p_2 w_1$

---

need to compare $p_1 w_2 \stackrel{?}{=} p_2 w_1$
dividing by $p_1 p_2$...

equivalent to comparing $w_2 / p_2 \stackrel{?}{=} w_1 / p_1$
$\omega_i = w_i / p_i$: Price per processing unit in dollars
Sort jobs in decreasing value of ω_i. Schedule jobs by this value.

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$\omega_i = w_i / p_i$: Price per processing unit in dollars
Sort jobs in decreasing value of ω_i. Schedule jobs by this value.

Correctness proof: Same as the unweighted case – if there is an inversion, then by the argument above, flip these jobs, and get a better schedule.

THE END

...

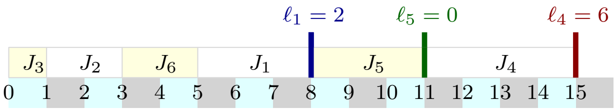
(for now)

19.4
Scheduling to Minimize Lateness

Scheduling to Minimize Lateness

1. Given jobs $J_1, J_2, \ldots, J_n$ with deadlines and processing times to be scheduled on a single resource.
2. If a job i starts at time s_i then it will finish at time $f_i = s_i + t_i$, where t_i is its processing time. d_i: deadline.
3. The lateness of a job is $\ell_i = \max(0, f_i - d_i)$.
4. Schedule all jobs such that $L = \max \ell_i$ is minimized.

	J_1	J_2	J_3	J_4	J_5	J_6
t_i	3	2	1	4	3	2
d_i	6	8	9	9	14	15



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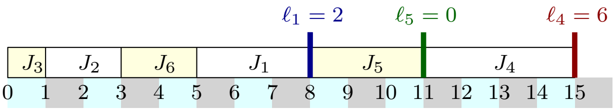

Greedy Template

```
Initially R is the set of all requests
curr_time = 0
max_lateness = 0
while R is not empty do
    choose i ∈ R
    curr_time = curr_time + t_i
    if (curr_time > d_i) then
        max_lateness = max(curr_time − d_i, max_lateness)

return max_lateness
```

Main task: Decide the order in which to process jobs in R

Greedy Template

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Main task: Decide the order in which to process jobs in R

Three Algorithms

1. Shortest job first — sort according to t_i.
2. Shortest slack first — sort according to $d_i - t_i$.
3. EDF = Earliest deadline first — sort according to d_i.

Counter examples for first two: exercise

Three Algorithms

1. Shortest job first — sort according to t_i.
2. Shortest slack first — sort according to $d_i - t_i$.
3. EDF = Earliest deadline first — sort according to d_i.

Counter examples for first two: exercise

Earliest Deadline First

Theorem 19.1.

Greedy with EDF rule minimizes maximum lateness.

Proof via an exchange argument.

Idle time: time during which machine is not working.

Lemma 19.2.

If there is a feasible schedule then there is one with no idle time before all jobs are finished.

Earliest Deadline First

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Lemma 19.2.
If there is a feasible schedule then there is one with no idle time before all jobs are finished.

Inversions

EDF = Earliest Deadline First

Assume jobs are sorted such that $d_1 \leq d_2 \leq \ldots \leq d_n$. Hence EDF schedules them in this order.

Definition 19.3.

A schedule S is said to have an inversion if there are jobs i and j such that S schedules i before j, but $d_i > d_j$.

Claim 19.4.

If a schedule S has an inversion then there is an inversion between two adjacent scheduled jobs.

Proof: exercise.

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Claim 19.4.

If a schedule S has an inversion then there is an inversion between two adjacent scheduled jobs.

Proof: exercise.

Proof sketch of Optimality of EDF

- Let S be an optimum schedule with smallest number of inversions.
- If S has no inversions then this is same as EDF and we are done.
- Else S has two adjacent jobs i and j with $d_i > d_j$.
- Swap positions of i and j to obtain a new schedule S'

Claim 19.5.

Maximum lateness of S' is no more than that of S. And S' has strictly fewer inversions than S.

THE END

...

(for now)

19.5
Maximum Weight Subset of Elements: Cardinality and Beyond

Picking k elements to maximize total weight

1. Given n items each with non-negative weights/profits and integer $1 \leq k \leq n$.
2. Goal: pick k elements to maximize total weight of items picked.

	e_1	e_2	e_3	e_4	e_5	e_6
weight	3	2	1	4	3	2

$k = 2$:
$k = 3$:
$k = 4$:

Greedy Template

```
N is the set of all elements X ← ∅
(* X will store all the elements that will be picked *)
while |X| < k and N is not empty do
    choose e_j ∈ N of maximum weight
    add e_j to X
    remove e_j from N
return the set X
```

Remark: One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top k elements but the above template generalizes to other settings a bit more easily.

Theorem 19.1.

Greedy is optimal for picking k elements of maximum weight.

Greedy Template

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Remark: One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top k elements but the above template generalizes to other settings a bit more easily.

Theorem 19.1.

Greedy is optimal for picking k elements of maximum weight.

A more interesting problem

1. Given n items $N = \{e_1, e_2, \ldots, e_n\}$. Each item e_i has a non-negative weight w_i.
2. Items partitioned into h sets $N_1, N_2, \ldots, N_h$. Think of each item having one of h colors.
3. Given integers $k_1, k_2, \ldots, k_h$ and another integer k
4. Goal: pick k elements such that no more than k_i from N_i to maximize total weight of items picked.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
weight	9	5	4	7	5	2	1

$N_1 = \{e_1, e_2, e_3\}$, $N_2 = \{e_4, e_5\}$, $N_3 = \{e_6, e_7\}$
$k = 4$, $k_1 = 2$, $k_2 = 1$, $k_3 = 2$

Greedy Template

```
N is the set of all elements X ← ∅
(* X will store all the elements that will be picked *)
while N is not empty do
    N' = {e_i ∈ N | X ∪ {e_i} is feasible}
    if N' = ∅ then break
    choose e_j ∈ N' of maximum weight
    add e_j to X
    remove e_j from N
return the set X
```

Theorem 19.2.

Greedy is optimal for the problem on previous slide.

Proof: exercise after class.

Special case of general phenomenon of Greedy working for maximum weight independent set in a matroid. Beyond scope of course.

Greedy Template

```
N is the set of all elements X ← ∅
(* X will store all the elements that will be picked *)
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return the set X
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THE END

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(for now)

19.6
Interval Scheduling

19.6.1
Problem statement, and a few greedy algorithms that do not work

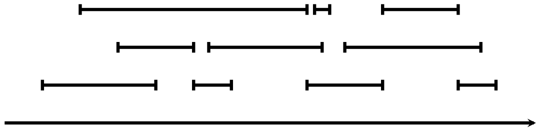
Interval Scheduling

Problem 19.1 (Interval Scheduling).

Input: *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

Goal: *Schedule as many jobs as possible*

1. *Two jobs with overlapping intervals cannot both be scheduled!*



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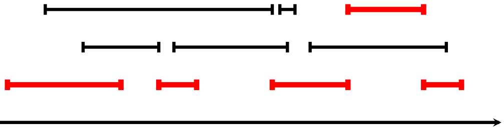

Greedy Template

```
R is the set of all requests
X ← ∅ (* X will store all the jobs that will be scheduled *)
while R is not empty do
    choose i ∈ R
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```

Main task: Decide the order in which to process requests in R

Greedy Template

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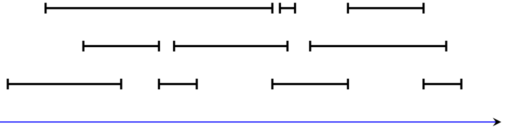
Main task: Decide the order in which to process requests in R

Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



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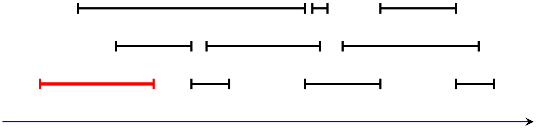

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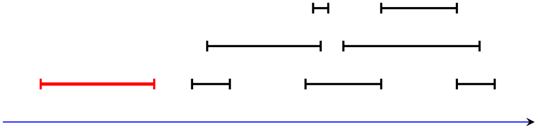

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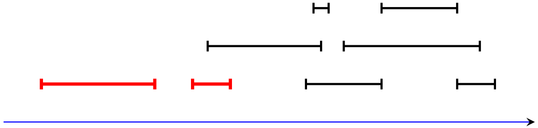

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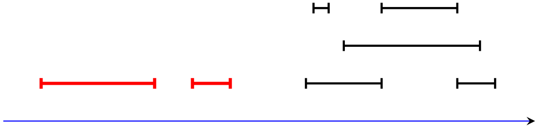

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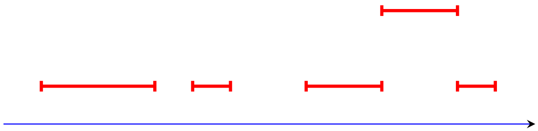

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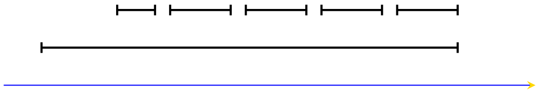

Figure: Counter example for earliest start time

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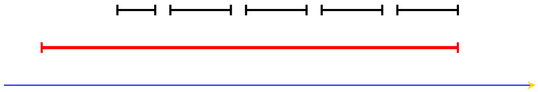

Figure: Counter example for earliest start time

Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.

Figure: Counter example for earliest start time

Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.

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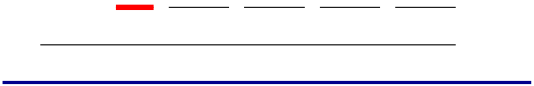

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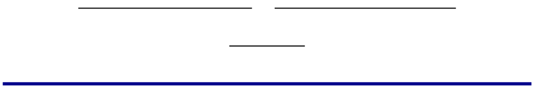

Figure: Counter example for smallest processing time

Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.

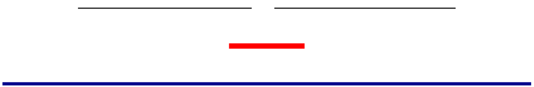

Figure: Counter example for smallest processing time

Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.

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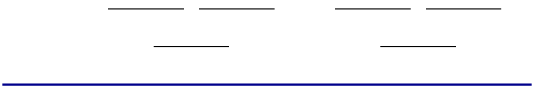
Fewest Conflicts

Process jobs in that have the fewest “conflicts” first.



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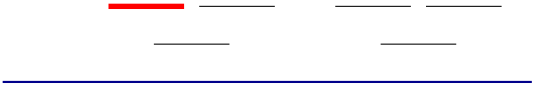

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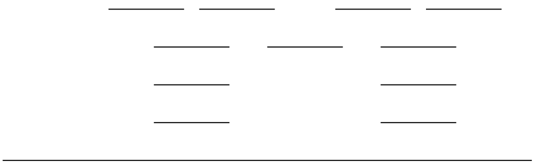

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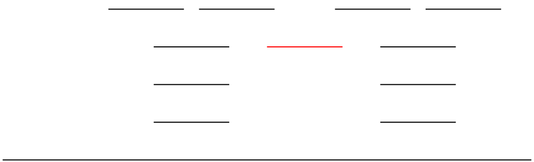

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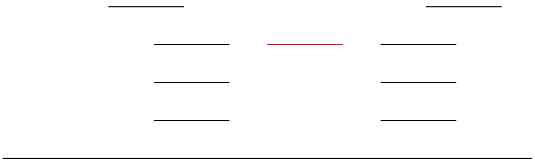

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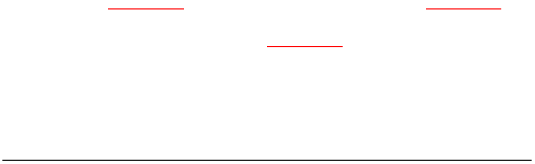

Figure: Counter example for fewest conflicts

THE END

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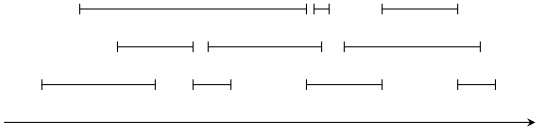
(for now)

19.6.2
Interval Scheduling: Earliest finish time

Earliest Finish Time

Process jobs in the order of their finishing times, beginning with those that finish earliest.



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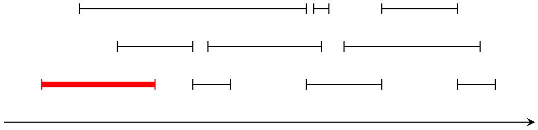

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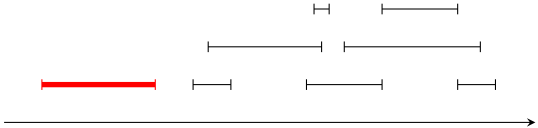

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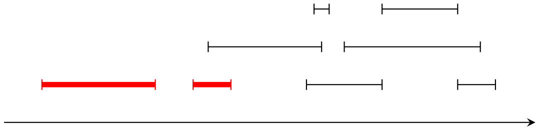

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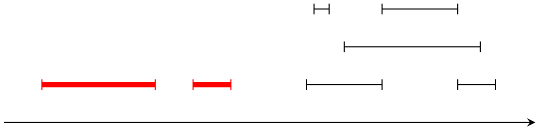

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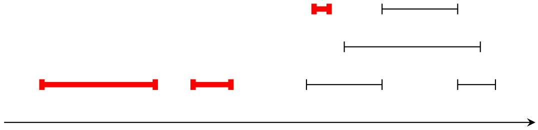

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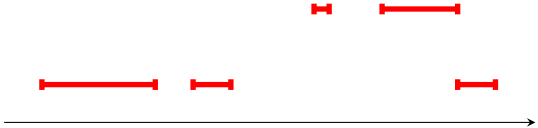

Optimal Greedy Algorithm

```
R is the set of all requests
X ← ∅ (* X stores the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is smallest
    add i to X
    remove from R all requests that overlap with i
return X
```

Theorem 19.2.

The greedy algorithm that picks jobs in the order of their finishing times is optimal.

Implementation and Running Time

```
Initially R is the set of all requests
X ← ∅ (* X stores the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    if i does not overlap with requests in X
        add i to X
    remove i from R
return the set X
```

- Presort all requests based on finishing time. $O(n \log n)$ time
- Now choosing least finishing time is $O(1)$
- Keep track of the finishing time of the last request added to A. Then check if starting time of i later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

Comments

1. Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
2. All requests need not be known at the beginning. Such online algorithms are a subject of research

Weighted Interval Scheduling

Suppose we are given n jobs. Each job i has a start time s_i, a finish time f_i, and a weight w_i. We would like to find a set S of compatible jobs whose total weight is maximized. Which of the following greedy algorithms finds the optimum schedule?

- Earliest start time first.
- Earliest finish time fist.
- Highest weight first.
- None of the above.
- IDK.

Weighted problem can be solved via dynamic programming. See notes.

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- Highest weight first.
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- IDK.

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THE END

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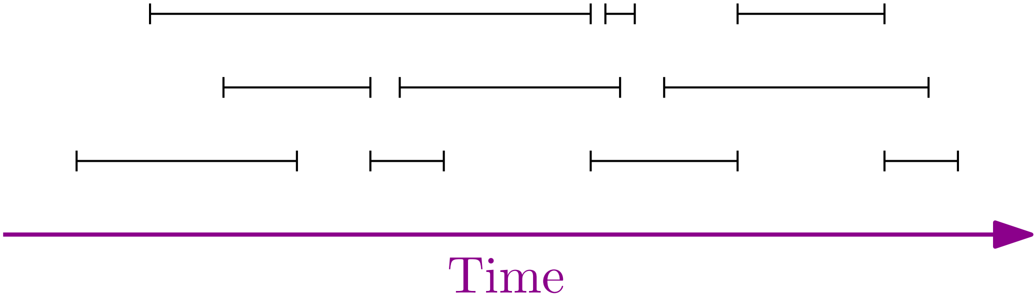
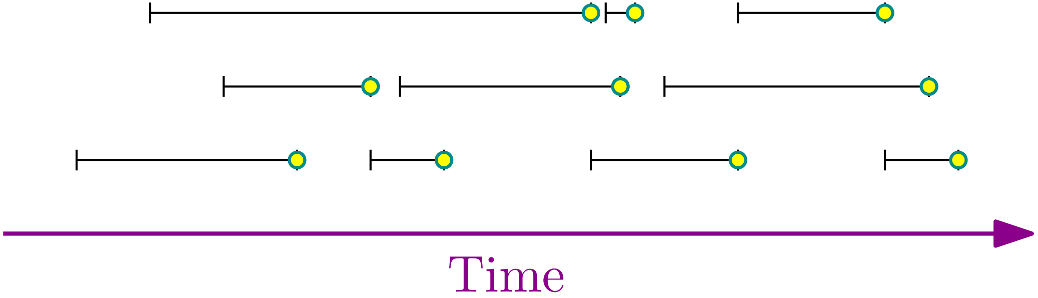
(for now)

19.6.3
Proving optimality of earliest finish time

Earliest finish time: A quick recall



Earliest finish time: A quick recall



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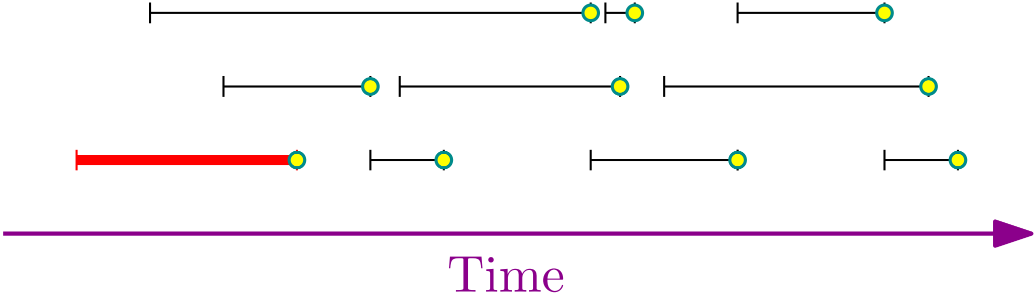

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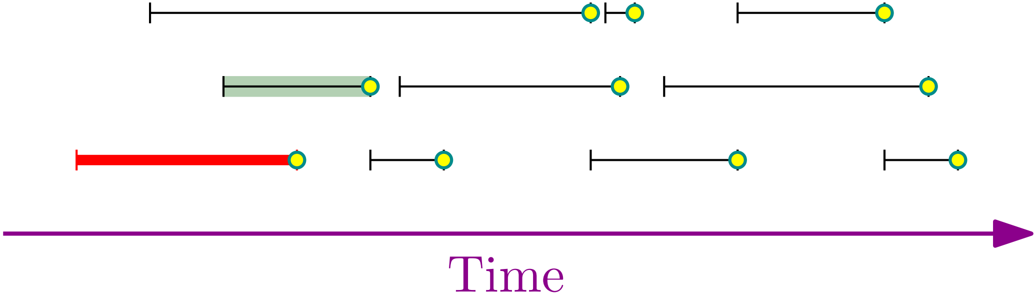

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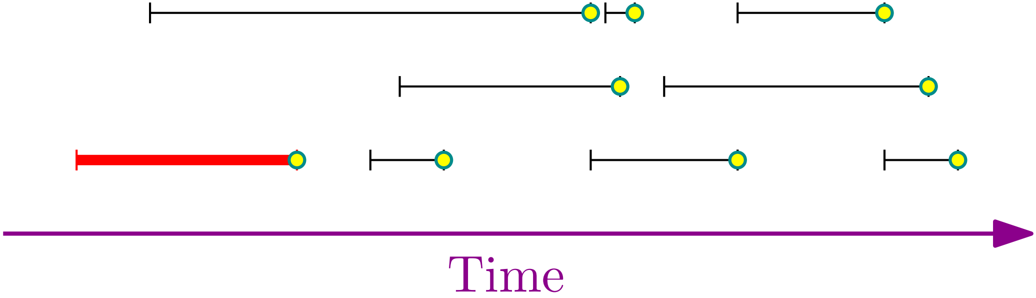

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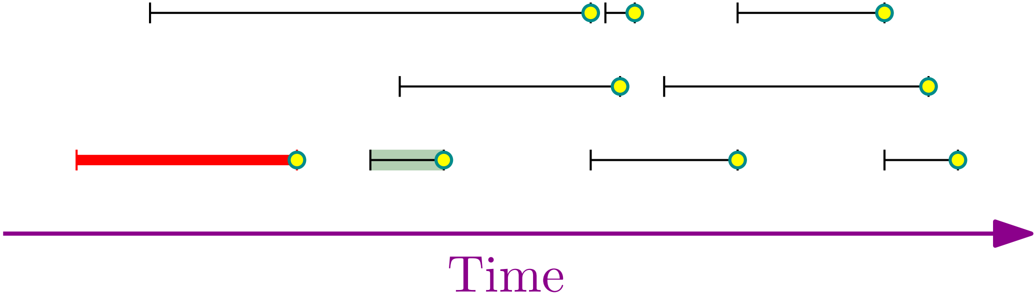

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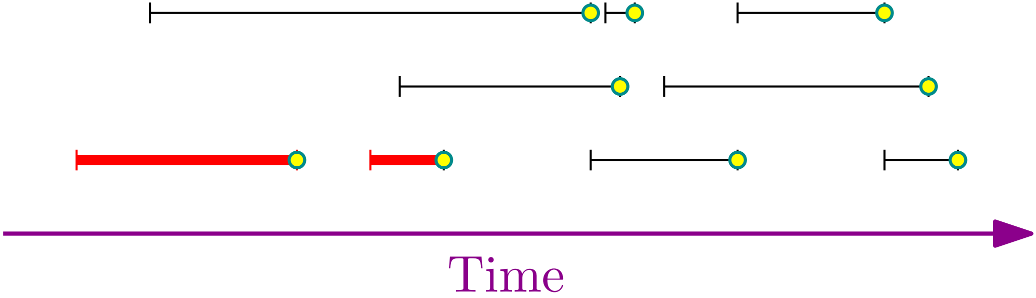

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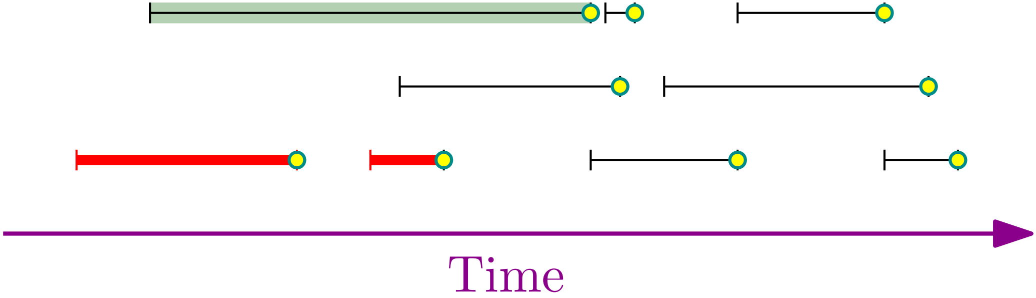

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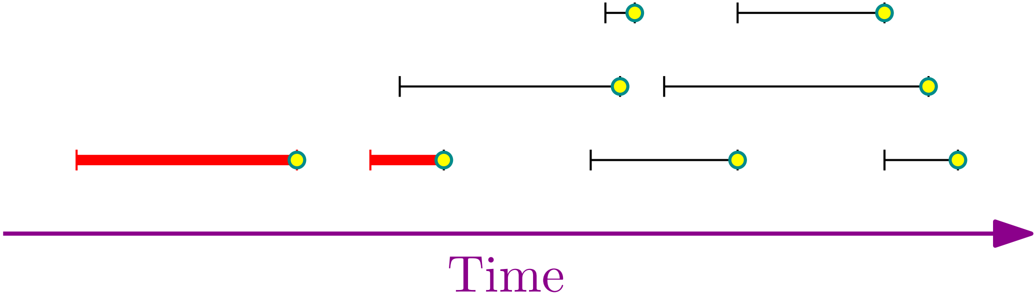

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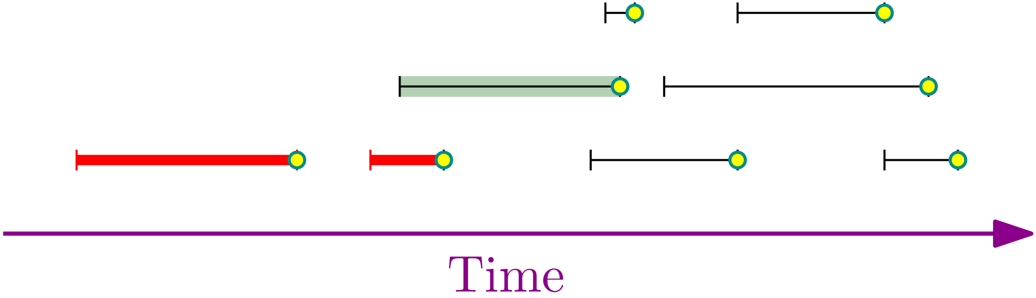

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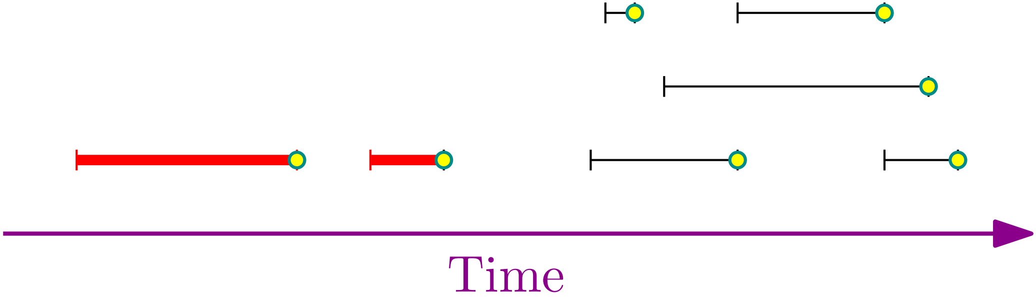

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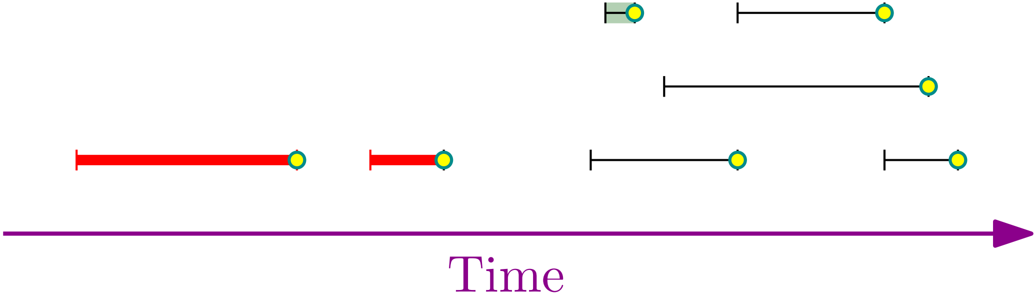

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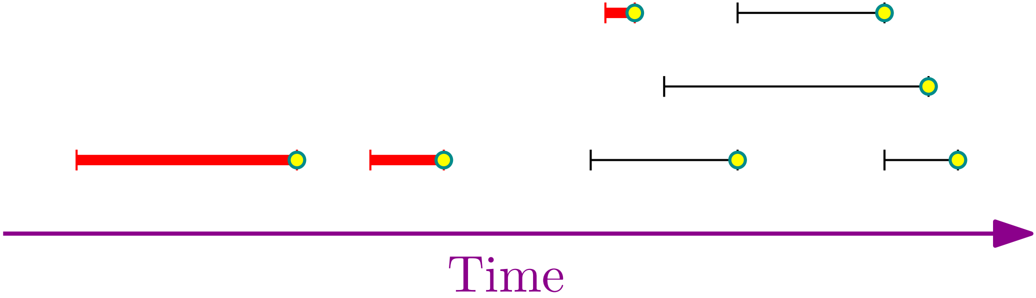

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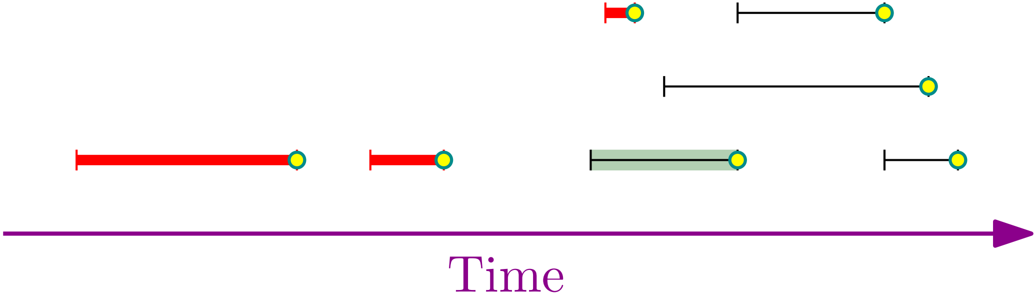

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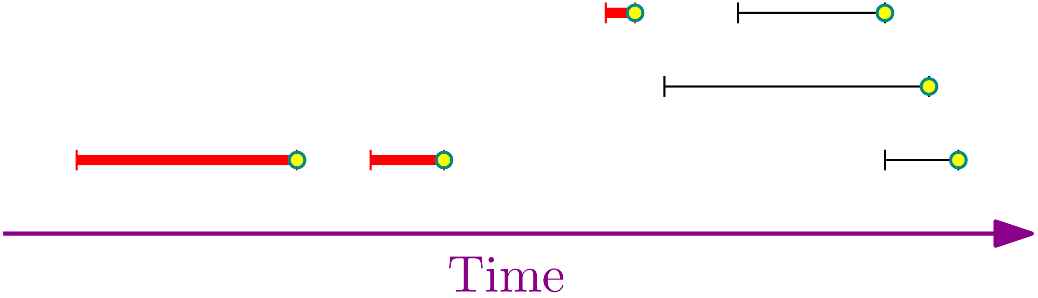

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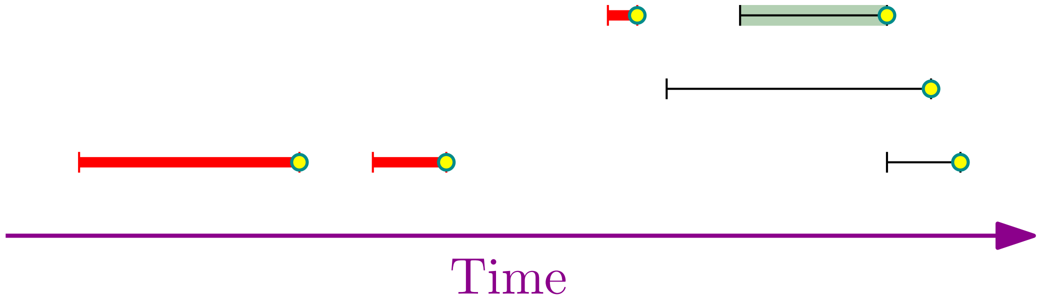

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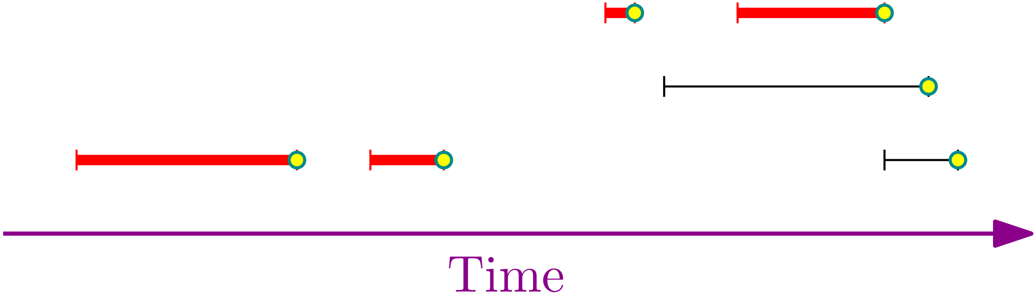

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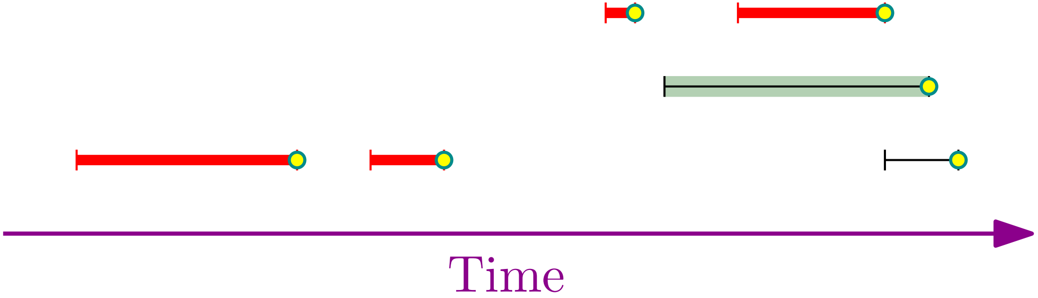

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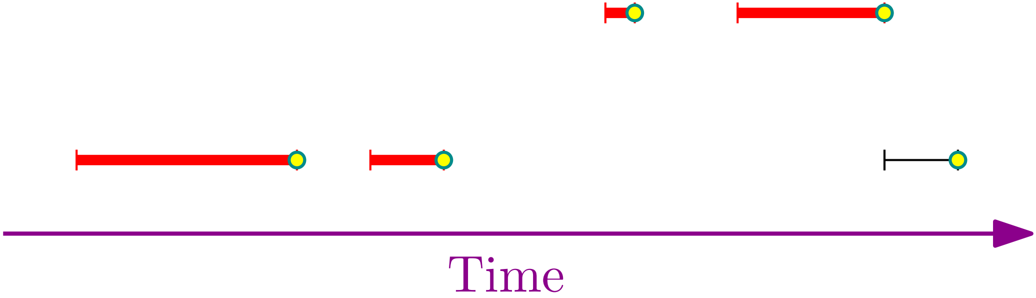

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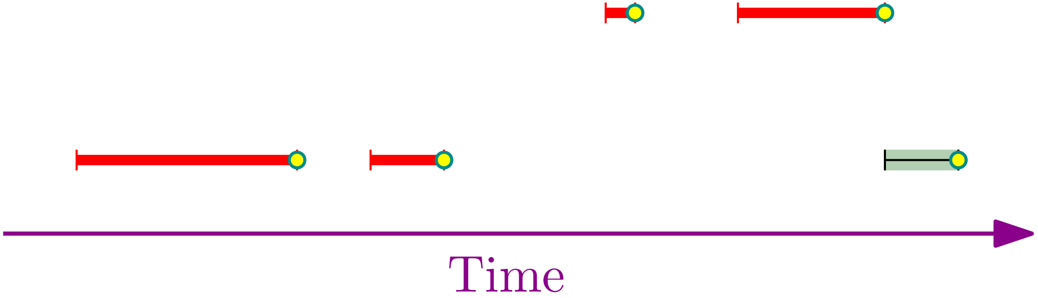

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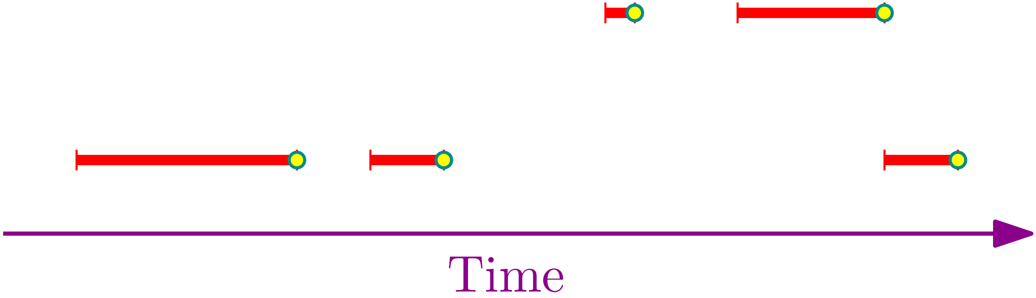

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1. Correctness: Clearly the algorithm returns a set of jobs that does not have any conflicts
2. For a set of requests R, let O be an optimal set and let X be the set returned by the greedy algorithm. Then $O = X$? Not likely!

   Instead we will show that $|O| = |X|$

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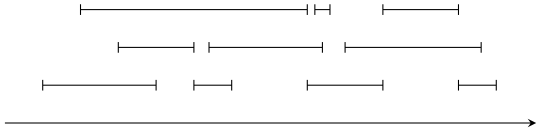

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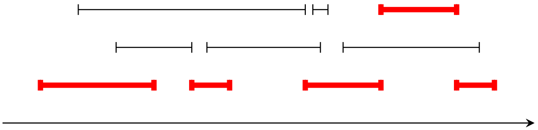

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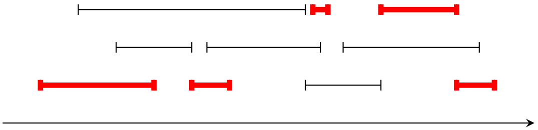

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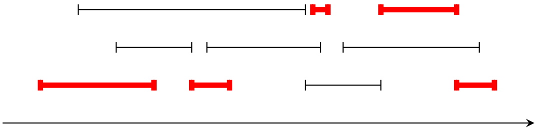

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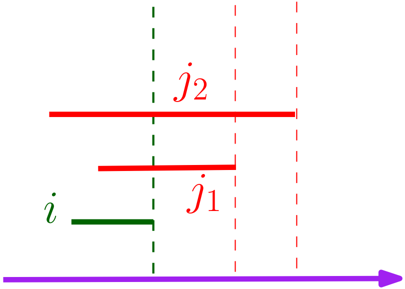
Helper Claim

Claim 19.3.

i be first interval picked by Greedy into solution.
O: Optimal solution.
If $i \notin O$, there is exactly one interval $j_1 \in O$ that conflicts with i.

Proof.

1. No $j \in O$ conflicts $i \implies O$ is not opt!
2. Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both j_1 and j_2 conflict with i.
3. Since i has earliest finish time, j_1 and i overlap at $f(i)$.
4. For same reason j_2 also overlaps with i at $f(i)$.
5. Implies that j_1, j_2 overlap at $f(i)$ but intervals in O cannot overlap. □



Proof of Optimality: Key Lemma

Lemma 19.4.

i_1 be first interval picked by Greedy. There exists an optimum solution that contains i_1.

Proof.

Let O be an arbitrary optimum solution. If $i_1 \in O$ we are done.

By **Claim 19.3** ...

1. Exists exactly one $j_1 \in O$ conflicting with i_1.
2. Form a new set O' by removing j_1 from O and adding i_1, that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
3. From claim, O' is a feasible solution (no conflicts).
4. Since $|O'| = |O|$, O' is also an optimum solution and it contains i_1. □

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Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: $n = 1$. Trivial since Greedy picks one interval.

Induction Step: Assume theorem holds for $i < n$.

Let K be an input (i.e., instance) with n intervals

$i_1 \Leftarrow$ First interval picked by greedy algorithm.

$K' \Leftarrow$ The result of removing i_1 and all conflicting intervals from K.

$|K'| = |K| - 1$.

$G(K), G(K')$: Solution produced by Greedy on K and K', respectively.

Lemma 19.4 $\implies$ optimum solution O to K with $i_1 \in O$.

Let $O' = O - \{i_1\}$. O' is a solution to K'.

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\begin{aligned}
|G(K)| &= 1 + |G(K')| && \text{from Greedy description} \\
&\geq 1 + |O'| && \text{By induction, } G(I') \text{ is optimum for } I') \\
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THE END

...

(for now)

19.7
Greedy algorithms – an epilogue

Greedy proof techniques: Overview

1. Greedy's first step leads to an optimum solution. Show that optimal solution can be modified to agree with greedy after first step. Then use induction. Example, Interval Scheduling.
2. Greedy algorithm stays ahead. Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
3. Structural property of solution. Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning (see Kleinberg-Tardos book).
4. Exchange argument. Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality.
   Example: Minimizing lateness, and Interval scheduling

Takeaway Points

1. Greedy algorithms come naturally but often are incorrect.
   A proof of correctness is an absolute necessity.
2. <u>Exchange</u> arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
3. Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.