CS/ECE 374 A ✧ Fall 2021
Homework 5

Due Tuesday, October 5, 2021 at 8pm Central Time

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1. Consider the following cruel and unusual sorting algorithm, proposed by Gary Miller:

```
CRUEL($A[1..n]$):
  if $n > 1$
      CRUEL($A[1..n/2]$)
      CRUEL($A[n/2+1..n]$)
      UNUSUAL($A[1..n]$)
```

```
UNUSUAL($A[1..n]$):
  if $n = 2$
      if $A[1] > A[2]$                    ⟨⟨the only comparison!⟩⟩
          swap $A[1] \longleftrightarrow A[2]$
  else
      for $i \leftarrow 1$ to $n/4$                ⟨⟨swap 2nd and 3rd quarters⟩⟩
          swap $A[i+n/4] \longleftrightarrow A[i+n/2]$
      UNUSUAL($A[1..n/2]$)                ⟨⟨recurse on left half⟩⟩
      UNUSUAL($A[n/2+1..n]$)              ⟨⟨recurse on right half⟩⟩
      UNUSUAL($A[n/4+1..3n/4]$)           ⟨⟨recurse on middle half⟩⟩
```

The comparisons performed by Miller's algorithm do not depend at all on the values in the input array; such a sorting algorithm is called ***oblivious***. Assume for this problem that the input size n is always a power of 2.

(a) Prove by induction that CRUEL correctly sorts any input array. *[Hint: Follow the smallest $n/4$ elements. Follow the largest $n/4$ elements. Follow the middle $n/2$ elements. What does UNUSUAL actually* ***do****??]*

(b) Prove that CRUEL would *not* correctly sort if we removed the for-loop from UNUSUAL.

(c) Prove that CRUEL would *not* correctly sort if we swapped the last two lines of UNUSUAL.

(d) What is the running time of UNUSUAL? Justify your answer.

(e) What is the running time of CRUEL? Justify your answer.

2. Dakshita is putting together a list of famous cryptographers, each with their dates of birth and death: al-Kindi (801–873), Giovanni Fontana (1395–1455), Leon Alberti (1404–1472), Charles Babbage (1791–1871), Alan Turing (1912–1954), Joan Clarke (1917–1996), Ann Caracristi (1921–2016), and so on. She wonders which two cryptographers on her list had the longest overlap between their lifetimes. For example, among the seven example cryptographers, Clarke and Caracristi had the longest overlap of 45 years (1921–1966).

Dakshita formalizes her problem as follows. The input is an array $A[1..n]$ of records, each with two numerical fields $A[i].\mathit{birth}$ and $A[i].\mathit{death}$ and a string field $A[i].\mathit{name}$. The desired output is the maximum, over all indices $i \neq j$, of the overlap length

$$\min\left\{A[i].\mathit{death},\ A[j].\mathit{death}\right\} - \max\left\{A[i].\mathit{birth},\ A[j].\mathit{birth}\right\}.$$

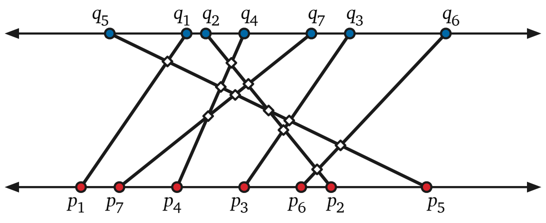
Describe and analyze an efficient algorithm to solve Dakshita's problem.

[Hint: Start by splitting the list in half by birth date. Do not *assume that cryptographers always die in the same order they are born. Assume that birth and death dates are distinct and accurate to the nanosecond.]*

Rubrics

Solved Problems

4. Suppose we are given two sets of n points, one set $\{p_1, p_2, \dots, p_n\}$ on the line $y = 0$ and the other set $\{q_1, q_2, \dots, q_n\}$ on the line $y = 1$. Consider the n line segments connecting each point p_i to the corresponding point q_i. Describe and analyze a divide-and-conquer algorithm to determine how many pairs of these line segments intersect, in $O(n \log n)$ time. See the example below.



Seven segments with endpoints on parallel lines, with 11 intersecting pairs.

Your input consists of two arrays $P[1\,..\,n]$ and $Q[1\,..\,n]$ of x-coordinates; you may assume that all $2n$ of these numbers are distinct. No proof of correctness is necessary, but you should justify the running time.

Solution: We begin by sorting the array $P[1\,..\,n]$ and permuting the array $Q[1\,..\,n]$ to maintain correspondence between endpoints, in $O(n \log n)$ time. Then for any indices $i < j$, segments i and j intersect if and only if $Q[i] > Q[j]$. Thus, our goal is to compute the number of pairs of indices $i < j$ such that $Q[i] > Q[j]$. Such a pair is called an ***inversion***.

We count the number of inversions in Q using the following extension of mergesort; as a side effect, this algorithm also sorts Q. If $n < 100$, we use brute force in $O(1)$ time. Otherwise:

- Color the elements in the Left half $Q[1\,..\,\lfloor n/2 \rfloor]$ bLue.
- Color the elements in the Right half $Q[\lfloor n/2 \rfloor + 1\,..\,n]$ Red.
- Recursively count inversions in (and sort) the blue subarray $Q[1\,..\,\lfloor n/2 \rfloor]$.
- Recursively count inversions in (and sort) the red subarray $Q[\lfloor n/2 \rfloor + 1\,..\,n]$.
- Count red/blue inversions as follows:
  - MERGE the sorted subarrays $Q[1\,..\,n/2]$ and $Q[n/2 + 1\,..\,n]$, maintaining the element colors.
  - For each blue element $Q[i]$ of the now-sorted array $Q[1\,..\,n]$, count the number of smaller red elements $Q[j]$.

The last substep can be performed in $O(n)$ time using a simple for-loop:

```
CountRedBlue($A[1..n]$):
    $count \leftarrow 0$
    $total \leftarrow 0$
    for $i \leftarrow 1$ to $n$
        if $A[i]$ is red
            $count \leftarrow count + 1$
        else
            $total \leftarrow total + count$
    return $total$
```

Merge and CountRedBlue each run in $O(n)$ time. Thus, the running time of our inversion-counting algorithm obeys the mergesort recurrence $T(n) = 2T(n/2) + O(n)$. (We can safely ignore the floors and ceilings in the recursive arguments.) We conclude that the overall running time of our algorithm is $O(n \log n)$, as required.

Rubric: This is enough for full credit.

In fact, we can execute the third merge-and-count step directly by modifying the Merge algorithm, without any need for "colors". Here changes to the standard Merge algorithm are indicated in red.

```
MergeAndCount($A[1..n], m$):
    $i \leftarrow 1$;  $j \leftarrow m + 1$;  $count \leftarrow 0$;  $total \leftarrow 0$
    for $k \leftarrow 1$ to $n$
        if $j > n$
            $B[k] \leftarrow A[i]$;  $i \leftarrow i + 1$;  $total \leftarrow total + count$
        else if $i > m$
            $B[k] \leftarrow A[j]$;  $j \leftarrow j + 1$;  $count \leftarrow count + 1$
        else if $A[i] < A[j]$
            $B[k] \leftarrow A[i]$;  $i \leftarrow i + 1$;  $total \leftarrow total + count$
        else
            $B[k] \leftarrow A[j]$;  $j \leftarrow j + 1$;  $count \leftarrow count + 1$
    for $k \leftarrow 1$ to $n$
        $A[k] \leftarrow B[k]$
    return $total$
```

We can further optimize MergeAndCount by observing that *count* is always equal to $j - m - 1$, so we don't need an additional variable. (Proof: Initially, $j = m + 1$ and $count = 0$, and we always increment j and *count* together.)

```
MergeAndCount2($A[1..n], m$):
  $i \gets 1$; $j \gets m+1$; $total \gets 0$
  for $k \gets 1$ to $n$
      if $j > n$
          $B[k] \gets A[i]$; $i \gets i+1$; $total \gets total + j - m - 1$
      else if $i > m$
          $B[k] \gets A[j]$; $j \gets j+1$
      else if $A[i] < A[j]$
          $B[k] \gets A[i]$; $i \gets i+1$; $total \gets total + j - m - 1$
      else
          $B[k] \gets A[j]$; $j \gets j+1$
  for $k \gets 1$ to $n$
      $A[k] \gets B[k]$
  return $total$
```

MergeAndCount2 still runs in $O(n)$ time, so the overall running time is still $O(n \log n)$, as required. ■

Rubric: 10 points = 2 for base case + 2 for divide (split and recurse) + 4 for conquer (merge and count) + 2 for time analysis. This is neither the only way to correctly describe this algorithm nor the only correct $O(n \log n)$-time algorithm. No proof of correctness is required.

Max 3 points for a correct $O(n^2)$-time algorithm.

Notice that each boxed algorithm is preceded by a clear English description of the task that algorithm performs—not how the algorithm works, but the relationship between its input and its output. Each English description is worth 25% of the credit for that algorithm (rounding to the nearest point). For example, the CountRedBlue algorithm is worth 4 points ("conquer"); the English description alone ("For each blue element $Q[i]$ of the now-sorted array $Q[1..n]$, count the number of smaller red elements $Q[j]$.") is worth 1 point.