CS/ECE 374 A ✧ Fall 2021

Homework 7

Due Tuesday, October 19, 2021 at 8pm Central Time

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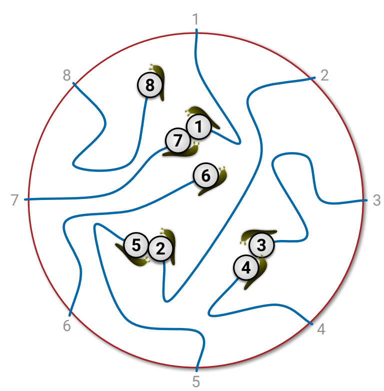
1. Every year, as part of its annual meeting, the Antarctican Snail Lovers of Upper Glacierville hold a Round Table Mating Race. Several high-quality breeding snails are placed at the edge of a round table. The snails are numbered in order around the table from 1 to n. During the race, each snail wanders around the table, leaving a trail of slime behind it. The snails have been specially trained never to fall off the edge of the table or to cross a slime trail, even their own. If two snails meet, they are declared a breeding pair, removed from the table, and whisked away to a romantic hole in the ground to make little baby snails. Note that some snails may never find a mate, even if the race goes on forever.



The end of a typical Antarctican SLUG race. Snails 6 and 8 never find mates.
The organizers must pay $M[3,4]+M[2,5]+M[1,7]$.

For every pair of snails, the Antarctican SLUG race organizers have posted a monetary reward, to be paid to the owners if that pair of snails meets during the Mating Race. Specifically, there is a two-dimensional array $M[1..n, 1..n]$ posted on the wall behind the Round Table, where $M[i,j] = M[j,i]$ is the reward to be paid if snails i and j Meet. Rewards may be positive, negative, or zero.

Describe and analyze an algorithm to compute the maximum total reward that the organizers could be forced to pay, given the array M as input.

2. Suppose you are given a NFA $M = (\{0, 1\}, Q, s, A, \delta)$ *without* ε-transitions and a binary string $w \in \{0, 1\}^*$. Describe and analyze an efficient algorithm to determine whether M accepts w. Concretely, the input NFA M is represented as follows:

   - $Q = \{1, 2, \ldots, k\}$ for some integer k.
   - The start state s is state 1.
   - Accepting states are represented by a boolean array $Acc[1..k]$, where $Acc[q] = \text{True}$ if and only if $q \in A$.
   - The transition function δ is represented by a boolean array $inDelta[1..k, 0..1, 1..k]$, where $inDelta[p, a, q] = \text{True}$ if and only if $q \in \delta(p, a)$.

   Your input consists of the integer k, the array $Acc[1..k]$, the array $inDelta[1..k, 0..1, 1..k]$, and the input string $w[1..n]$. Your algorithm should return True if M accepts w, and False if M does not accept w. Report the running time of your algorithm as a function of k (the number of states in M) and n (the length of w). *[Hint: Do not convert M to a DFA!!]*

Solved Problems

3. A string w of parentheses (and) and brackets [and] is ***balanced*** if and only if w is generated by the following context-free grammar:

$$S \to \varepsilon \mid (S) \mid [S] \mid SS$$

For example, the string $w =$ ([()][]())[()()]() is balanced, because $w = xy$, where

$$x = ([()][]()) \qquad \text{and} \qquad y = [()()]().$$

Describe and analyze an algorithm to compute the length of a longest balanced subsequence of a given string of parentheses and brackets. Your input is an array $A[1..n]$, where $A[i] \in \{(,),[,]\}$ for every index i.

Solution: Suppose $A[1..n]$ is the input string. For all indices i and k, let ***LBS(i, k)*** denote the length of the longest balanced subsequence of the substring $A[i..k]$. We need to compute $LBS(1, n)$. This function obeys the following recurrence:

$$LBS(i,j) = \begin{cases} 0 & \text{if } i \geq k \\ \max\left\{ \begin{array}{c} 2 + LBS(i+1,k-1) \\ \max\limits_{j=1}^{k-1} \big(LBS(i,j) + LBS(j+1,k)\big) \end{array} \right\} & \text{if } A[i] \sim A[k] \\ \max\limits_{j=1}^{k-1} \big(LBS(i,j) + LBS(j+1,k)\big) & \text{otherwise} \end{cases}$$

Here $\boldsymbol{A[i] \sim A[k]}$ indicates that $A[i]$ is a left delimiter and $A[k]$ is the corresponding right delimiter: Either $A[i] =$ (and $A[k] =$), or $A[i] =$ [and $A[k] =$].

We can memoize this function into a two-dimensional array $LBS[1..n, 1..n]$. Because each entry $LBS[i, j]$ depends only on entries in later rows or earlier columns (or both), we can evaluate this array row-by-row from bottom up in the outer loop, scanning each row from left to right in the inner loop. The resulting algorithm runs in $\boldsymbol{O(n^3)}$ ***time***.

```
LongestBalancedSubsequence(A[1..n]):
  for i ← n down to 1
      LBS[i,i] ← 0
      for k ← i+1 to n
          if A[i] ~ A[k]
              LBS[i,k] ← LBS[i+1,k−1] + 2
          else
              LBS[i,k] ← 0
          for j ← i to k−1
              LBS[i,k] ← max{LBS[i,k], LBS[i,j] + LBS[j+1,k]}
  return LBS[1,n]
```

■

Rubric: 10 points, standard dynamic programming rubric

4. Oh, no! You've just been appointed as the new organizer of Giggle, Inc.'s annual mandatory holiday party! The employees at Giggle are organized into a strict hierarchy, that is, a tree with the company president at the root. The all-knowing oracles in Human Resources have assigned a real number to each employee measuring how "fun" the employee is. In order to keep things social, there is one restriction on the guest list: An employee cannot attend the party if their immediate supervisor is also present. On the other hand, the president of the company *must* attend the party, even though she has a negative fun rating; it's her company, after all.

   Describe an algorithm that makes a guest list for the party that maximizes the sum of the "fun" ratings of the guests. The input to your algorithm is a rooted tree T describing the company hierarchy, where each node v has a field $v.\mathit{fun}$ storing the "fun" rating of the corresponding employee.

Solution (two functions): We define two functions over the nodes of T.

- $\mathit{MaxFunYes}(v)$ is the maximum total "fun" of a legal party among the descendants of v, where v is definitely invited.
- $\mathit{MaxFunNo}(v)$ is the maximum total "fun" of a legal party among the descendants of v, where v is definitely not invited.

We need to compute $\mathit{MaxFunYes}(\mathit{root})$. These two functions obey the following mutual recurrences:

$$\mathit{MaxFunYes}(v) = v.\mathit{fun} + \sum_{\text{children } w \text{ of } v} \mathit{MaxFunNo}(w)$$

$$\mathit{MaxFunNo}(v) = \sum_{\text{children } w \text{ of } v} \max\{\mathit{MaxFunYes}(w), \mathit{MaxFunNo}(w)\}$$

(These recurrences do not require separate base cases, because $\sum \varnothing = 0$.) We can memoize these functions by adding two additional fields $v.\mathit{yes}$ and $v.\mathit{no}$ to each node v in the tree. The values at each node depend only on the vales at its children, so we can compute all $2n$ values using a postorder traversal of T.

```
BestParty(T):
  ComputeMaxFun(T.root)
  return T.root.yes
```

```
ComputeMaxFun(v):
  v.yes ← v.fun
  v.no ← 0
  for all children w of v
      ComputeMaxFun(w)
      v.yes ← v.yes + w.no
      v.no ← v.no + max{w.yes, w.no}
```

(Yes, this is still dynamic programming; we're only traversing the tree recursively because that's the most natural way to traverse trees![a]) The algorithm spends $O(1)$ time at each node, and therefore runs in $\boldsymbol{O(n)}$ ***time*** altogether. ■

[a] A naïve recursive implementation would run in $O(\phi^n)$ time in the worst case, where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio. The worst-case tree is a path—every non-leaf node has exactly one child.

Solution (one function): For each node v in the input tree T, let $MaxFun(v)$ denote the maximum total "fun" of a legal party among the descendants of v, where v may or may not be invited.

The president of the company must be invited, so none of the president's "children" in T can be invited. Thus, the value we need to compute is

$$root.fun + \sum_{\text{grandchildren } w \text{ of } root} MaxFun(w).$$

The function $MaxFun$ obeys the following recurrence:

$$MaxFun(v) = \max \left\{ \begin{array}{l} v.fun + \displaystyle\sum_{\text{grandchildren } x \text{ of } v} MaxFun(x) \\ \displaystyle\sum_{\text{children } w \text{ of } v} MaxFun(w) \end{array} \right\}$$

(This recurrence does not require a separate base case, because $\sum\varnothing = 0$.) We can memoize this function by adding an additional field $v.maxFun$ to each node v in the tree. The value at each node depends only on the values at its children and grandchildren, so we can compute all values using a postorder traversal of T.

```
BestParty(T):
  ComputeMaxFun(T.root)
  party ← T.root.fun
  for all children w of T.root
      for all children x of w
          party ← party + x.maxFun
  return party
```

```
ComputeMaxFun(v):
  yes ← v.fun
  no ← 0
  for all children w of v
      ComputeMaxFun(w)
      no ← no + w.maxFun
      for all children x of w
          yes ← yes + x.maxFun
  v.maxFun ← max{yes, no}
```

(Yes, this is still dynamic programming; we're only traversing the tree recursively because that's the most natural way to traverse trees![a])

The algorithm spends $O(1)$ time at each node (because each node has exactly one parent and one grandparent) and therefore runs in ***$O(n)$ time*** altogether. ■

[a] Like the previous solution, a direct recursive implementation would run in $O(\phi^n)$ time in the worst case, where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio.

Rubric: 10 points: standard dynamic programming rubric. These are not the only correct solutions.