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## CS 374 LAB 24: REDUCTIONS

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**Problem 1.** [Category: Proof] A CNF formula  $\varphi$  is in  $k$ -CNF form if each clause of  $\varphi$  has exactly  $k$  literals. The  $k$ -SAT problem is to decide if a given  $k$ -CNF formula is satisfiable. In this problem we will reduce 3-SAT to 4-SAT. Note that we have seen in lecture how to reduce 4-SAT (and more generally SAT) to 3-SAT.

1. Suppose  $\varphi$  is a 3-CNF formula. Consider the following reduction where we add a new variable  $u$  and replace each clause  $c = (\ell_1 \vee \ell_2 \vee \ell_3)$  by a new clause  $c' = (\ell_1 \vee \ell_2 \vee \ell_3 \vee u)$ . Note that we are using the same variable for each clause. Let  $\varphi'$  be the new formula obtained from  $\varphi$  via this reduction. Prove that  $\varphi'$  is satisfiable if  $\varphi$  is satisfiable. Give an example to show that  $\varphi'$  is satisfiable but  $\varphi$  is not satisfiable.
2. Obtain a correct reduction by altering the preceding one and prove that  $\varphi'$  is satisfiable if and only if  $\varphi$  is satisfiable.

**Problem 2.** [Category: Proof] A path  $P$  in a directed graph  $G$  is called a Hamiltonian path if it contains all the vertices of  $G$ . The Hamiltonian Path problem is the following: given  $G$ , does  $G$  contain a Hamiltonian path? The Longest  $s$ - $t$  Path problem is the following: given a directed graph  $G = (V, E)$  two nodes  $s, t \in V$  and an integer  $k$ , is there a simple path of length at least  $k$  from  $s$  to  $t$  in  $G$ ?

1. Assuming that you have a black box algorithm for the Longest  $s$ - $t$  Path problem describe a polynomial-time algorithm for the Hamiltonian Path problem.
2. Did you use a mapping reduction for the preceding part? If not, give a mapping reduction from the Hamiltonian Path problem to the Longest  $s$ - $t$  Path problem. That is, given  $G$  your reduction should output a graph  $G' = (V', E')$ , two nodes  $s, t \in V'$  and an integer  $k$  such that  $G'$  has an  $s$ - $t$  simple path of length at least  $k$  if and only if  $G$  has a Hamiltonian Path.

**Problem 3.** [Category: Comprehension+Proof] **Self-reduction.** We focus on decision problems even when the underlying problem we are interested in is an optimization problem. For most problems of interest we can in fact show that a polynomial-time algorithm for the decision problem also implies a polynomial-time algorithm for the corresponding optimization problem. To illustrate this consider the maximum independent set (MIS) problem.

1. Suppose you are given a algorithm that given a graph  $H$  and integer  $\ell$  outputs whether  $H$  has an independent set of size at least  $\ell$ . Using this algorithm as a *black box*, describe a polynomial time algorithm that given a graph  $G$  and integer  $k$  outputs an independent set of size  $k$  in  $G$  if it has one. Note that you can use the black box algorithm more than once. *Hint:* What happens if you remove a vertex  $v$  and the independent set size does not decrease? What if it does?
2. How would you efficiently find a maximum independent set in a given graph  $G$  using the black box?