April 11, 2019 10:59 AM

Fall course announcements:

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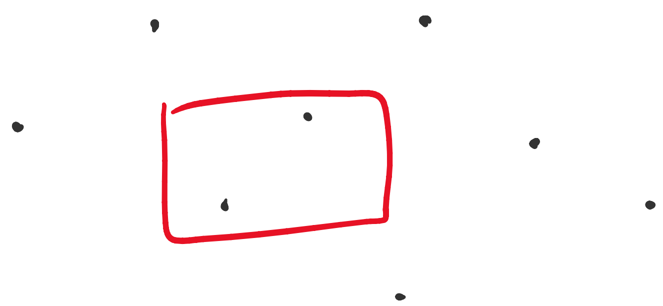
me CS598 Geometric Data Structures



PART III: Undecidability & NP-Completeness

how to prove problem is hard to solve...

Def L is <u>decidable</u>
if $\exists$ TM/program M that
halts on all input
& accepts w if $w \in L$
rejects w if $w \notin L$

Turing's Thm

$$\text{TM-Halt} = \{ \overbrace{\langle M, w \rangle}^{\text{encoding of } M, w} \mid \text{TM } M \text{ halts on input string } w \}$$

is undecidable.

Similarly,

Similarly,

TM-Acc = $\{ \langle M, w \rangle \mid$ TM M accepts input string $w \}$

is undecidable.

Pf: By contradiction.

Assume TM-Acc is decidable, by TM/alg'm M_{acc}.

Will construct a counterex M_{bad}, w_{bad}:

M_{bad} is this program:

on input $\langle M \rangle$,

run M_{acc} on $\langle M, \langle M \rangle \rangle$

if M_{acc} accepts, reject
else accept

$w_{bad} = \langle M_{bad} \rangle$.

Case 1. M_{acc} accepts $\langle M_{bad}, \langle M_{bad} \rangle \rangle$.

M_{bad} rejects $\langle M_{bad} \rangle$: wrong!

Case 2. M_{acc} rejects $\langle M_{bad}, \langle M_{bad} \rangle \rangle$

M_{bad} accepts : wrong! $\square$

From one hard problem, can prove other problems hard by reduction.

Ex 1. TM-Acc-All = $\{ \langle M \rangle \mid$ TM M accepts all inputs $\}$

$= \{ \langle M \rangle \mid L(M) = \Sigma^* \}$.

is undecidable.

[appl: main (int n):
  if n is even then accept
  if sum of all divisors of n is 2n
  then reject
  else accept]

OPEN!

Pf: By contradiction.

Assume TM-Acc-All is decidable by alg'm M_{all}.

Will give an alg'm M_{acc} to decide TM-Acc:

On input $\langle M, w \rangle$,

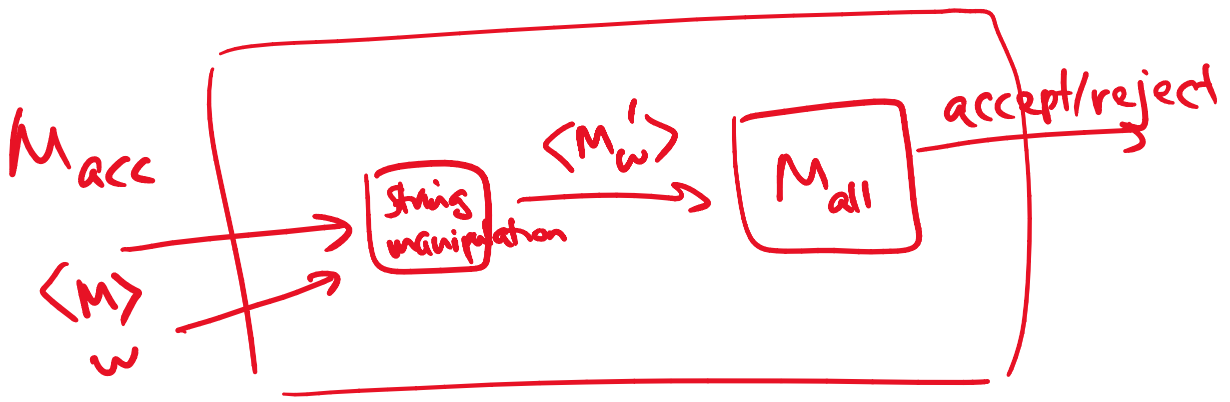
construct $\langle M'_w \rangle$ encoding of a new TM M'_w,

where on input x,
M'_w ignores x & just simulate M on w

i.e. given string M = " f(...) { ... } "
and string w,
construct new string

M'_w = " f(...) { ... }
main(string x) {
return f("w") } "

String manipulation: linear time



Then run M_{all} on $\langle M_w' \rangle$
accept iff M_{all} accepts.

Correctness:

M_{acc} accepts $\langle M, w \rangle$

$\iff$ M_{all} accepts $\langle M_w' \rangle$

$\iff$ M_w' accepts all input

$\iff$ M accepts w.

$\Rightarrow$ TM-Acc is decidable: Contradiction! $\square$

Ex2 TM-Acc-Some $= \{ \langle M \rangle \mid L(M) \neq \emptyset \}$
is undecidable.

Same proof as Ex 1!

TM-Empty $= \{ \langle M \rangle \mid L(M) = \emptyset \}$
is undecidable
(by closure under complement)

Ex3 TM-EQUIV $= \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$
is undecidable.

Pf: By contradiction.
Assume TM-EQUIV is decidable, by alg'm M_{equiv}.
Will give alg'm to decide TM-Acc-All:

On input $\langle M \rangle$,
create TM M_2 with $L(M_2) = \Sigma^*$
just run M_{equiv} on $\langle M, M_2 \rangle$.

just run M_{equiv} on $\langle M, M_2 \rangle$. □

Ex4 $\text{TM-Reg} = \{ \langle M \rangle \mid L(M) \text{ is regular for TM } M \}$

is undecidable.

Pf: Very similar to Ex 1.

By contradiction.

Assume TM-Reg is decidable, by alg'm M_{reg}.

Will give an alg'm M_{acc} to decide TM-Acc as follows:

On input $\langle M, w \rangle$,

1. construct encoding $\langle M_w' \rangle$ of a new TM M_w' where on input x,
   it simulates M on w
   if M accepts w
   then M_w' accepts x iff x is a palindrome
   else reject

string manipulation

2. run M_{reg} on $\langle M_w' \rangle$
3. accept iff M_{reg} rejects

Correctness:

a non-regular lang

$$L(M_w') = \begin{cases} \{x \mid x \text{ is a palindrome}\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{if } M \text{ rejects } w \text{ or } M \text{ does not halts on } w \end{cases}$$

a reg. lang.

M_{acc} accepts $\langle M, w \rangle$

$\iff$ M_{reg} rejects $\langle M'_w \rangle$

$\iff$ $L(M'_w)$ is not regular

$\iff$ M accepts w.

So, TM-Acc is decidable : Contradiction! $\square$

Rmk: proof very general

<u>Rice's Thm</u> Let $\mathcal{P}$ be a property about langs.

$\{ \langle M \rangle \mid L(M) \text{ has property } \mathcal{P} \}$

is undecidable

if $\mathcal{P}$ is <u>nontrivial</u>

(i.e. some decidable lang. has $\mathcal{P}$
& some " " does not have $\mathcal{P}$)

<u>Ex 5</u> $\{ \langle M \rangle \mid M \text{ accepts exactly } 374 \text{ strings} \}$
undecidable by Rice

⋮

$\{ \langle M \rangle \mid M \text{ runs in} \leq 100 \text{ steps} \}$
Rice <u>not</u> applicable!

Other undecidable problems

a) $\{ \langle G \rangle \mid L(G) = \Sigma^* \text{ for CFG } G \}$

b) Hilbert's 10th Problem:

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does a polynomial eq'n has integer sol'ns?

$$x^7 + y^7 = z^7 + 3$$

c) tiling

$\vdots$