April 16, 2019 10:59 AM

Announcement:

HW 11 (keep 21 of 33 scores)

Last Time:

Prove undecidability

What about problems that are decidable but have no efficient alg'ms?

polynomial-time

Similar Plan:

- start with one hard problem
- Prove other problems hard by poly-time reduction

Def P = class of all langs/decision problems that can be solved in poly. time

Main Def Given decision problems L_1, L_2, a poly-time reduction from L_1 to L_2 is an poly-time alg'm f s.t. $\forall$ input x,

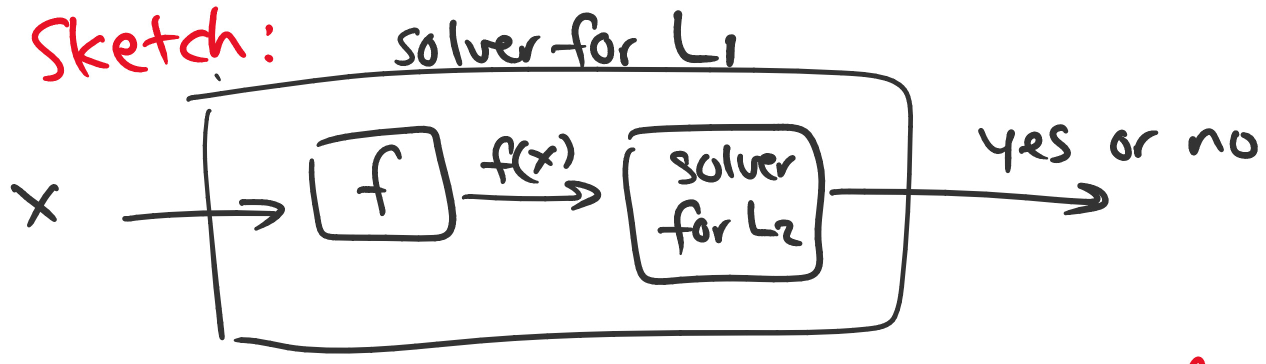
Karp reduction

$$\boxed{\text{output of } L_1 \text{ on } x \text{ is yes} \iff \text{output of } L_2 \text{ on } f(x) \text{ is yes}}$$

Notation Write $L_1 \le_p L_2$.

Fact 1 If $L_1 \le_p L_2$ and $L_2 \in P$, then $L_1 \in P$.

Pf Sketch:



(note: composition of two polynomials is polynomial)

Contrapositive: If $L_1 \le_p L_2$ and $L_1 \notin P$, then $L_2 \notin P$.

Fact 2 If $L_1 \le_p L_2$ and $L_2 \le_p L_3$, then $L_1 \le_p L_3$.

Pf: By composition. $\square$

Examples of Reduction

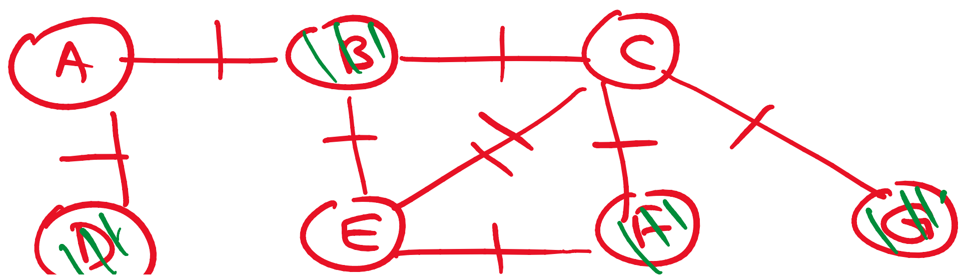
Ex 1 Vertex Cover: (decision vers.)

Input: undir graph $G=(V,E)$, integer k

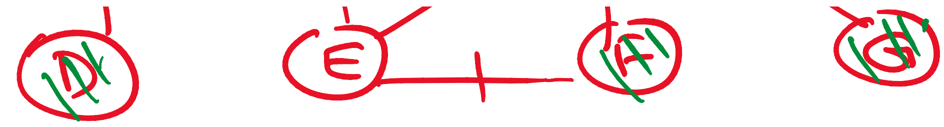
Output: yes iff $\exists$ vertex cover of size $\le k$

iff $\exists$ subset $S \subseteq V$ of size $\le k$

s.t. $\forall uv \in E \Rightarrow u \in S$ or $v \in S$



indep set
$\{B, D, F, G\}$
$k=4$



$k=4$

VC: $\{A, C, E\}$ $\quad k=3$

Set-Cover:

Input: set U, $A_1, \ldots, A_n \subseteq U$, integer k

Output: yes iff $\exists\, I \subseteq \{1, \ldots, n\}$,

s.t. $\bigcup_{i \in I} A_i = U$.

e.g. $U = \{1,2,3,4,5\}$

$\{1,3,5\}$, $\underline{\{2,5\}}$, $\underline{\{1,2,3\}}$, $\underline{\{3,4\}}$

$k=3$

Vertex-Cover $\leq_p$ Set-Cover:

Pf: Given input to Vertex-Cover: $G=(V,E)$, k, $\quad V=\{v_1, \ldots, v_n\}$

construct input to Set-Cover: $\underline{U, A_1, \ldots, A_n, k'}$

where $U = E$, $\quad k' = k$

$A_i = \{ e \in E \mid e \text{ is incident to } v_i \}$

This construction takes poly time.

Correctness: $\exists$ vertex cover S of size $\leq k$ in G

$\iff \exists\, I \subseteq \{1, \ldots, n\}$ of size $\leq k$

s.t. $\bigcup_{i \in I} A_i = E$.

Ex 2 Independent Set: (decision vers)

Input: undir graph $G=(V,E)$, integer k

Output: yes iff $\exists$ indep set S of size $\geq k$

$S \subseteq V$ s.t.

$S \subseteq V$ s.t.

$\forall u,v \in S \Rightarrow uv \notin E$

i.e. $\forall\, uv \in E \Rightarrow u \notin S$ or $v \notin S$

Indep Set $\leq_p$ Vertex Cover: i.e. $V-S$ is a vertex cover

Pf: Given input to Indep Set: $G=(V,E)$, k

construct input to Vertex Cover: G', k'

polytime

where $G' = G$

$k' = n-k$

Correctness: $\exists$ indep set S of size $\geq k$

$\iff$ $\exists$ vertex cover S' of size $\leq n-k$.

$(S' = V-S)$ $\square$

Vertex-Cover $\leq_p$ Indep-Set:

Same

How to get first hard problem??

Def NP = class of all decision problems of the form

Input: x

Output: yes iff

$\exists y$ s.t. $C(x,y)$ is true

certificate

where ① y has poly size

② C runs in poly time

certifier / verifier

Ex Vertex-Cover $\in$ NP:

certificate: subset $S \subseteq V$ ← poly size

certifier: check $|S| \leq k$

& $\forall uv \in E \Rightarrow u \in S$ or $v \in S$

poly time

Set-Cover $\in$ NP.

Indep-Set $\in$ NP.

⋮

Rmk: NP stands for Nondeterministic Polytime

idea - the "hardest" problem in NP ??