April 23, 2019 10:59 AM

Last Time:

- SAT is NP-complete.
- 3SAT is NP-complete.

Recipe: To show L is NP-complete:

1. $L \in NP$
2. $L_0 \le_p L$ for some known NP-complete L_0.

Independent Set

Input: $G=(V,E)$, integer k

Output: yes iff $\exists$ indep sets of size $(\geq)$ k.

Thm Indep-Set is NP-complete.

Pf: ① Indep-Set $\in$ NP:

Certificate: subset $S \subseteq V$. } poly size

Certifier: checks $|S| \geq k$ & $\forall u,v \in S$, $uv \notin E$. } poly time

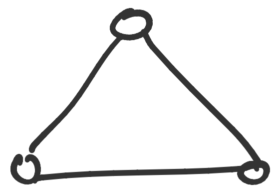
② $3SAT \le_p$ Indep-Set:

Given input to 3SAT: a 3CNF formula F with n vars, m clauses,

construct input to Indep Set: graph $G=(V,E)$, & integer k construct input to Indep Set: graph $G=(V,E)$, & integer k.

as follows:

for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$,
create 3 vertices v_{i1}, v_{i2}, v_{i3}
& 3 edges $v_{i1}v_{i2}$, $v_{i2}v_{i3}$, $v_{i3}v_{i1}$



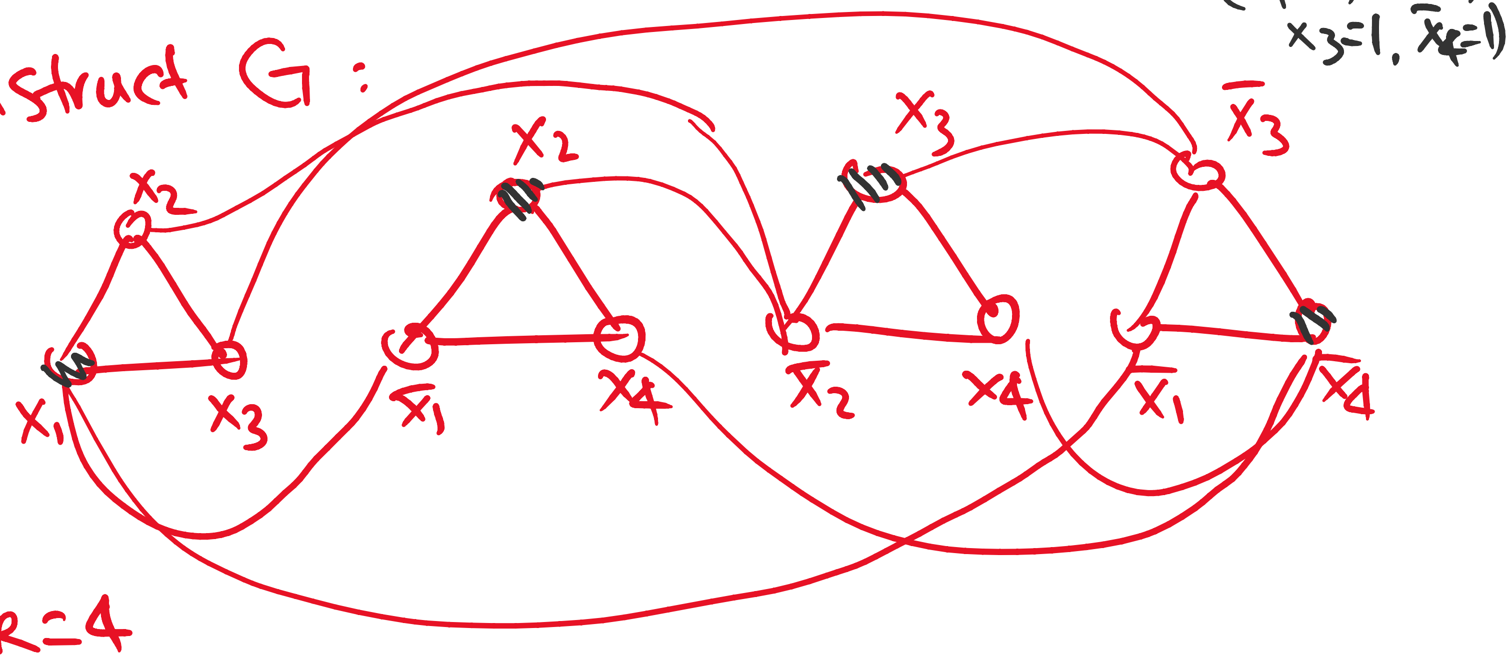
whenever $\alpha_{ij} = \overline{\alpha_{i'j'}}$,
add edge $v_{ij} v_{i'j'}$ ← "cross" edges

Set $k = m$.
Construction $F \to (G, k)$ takes poly time.
($O(m)$ vertices in G, $O(m^2)$ edges).

e.g. given $F = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4) \wedge (\overline{x}_2 \vee x_3 \vee x_4) \wedge (\overline{x}_1 \vee \overline{x}_3 \vee \overline{x}_4)$

$(x_1=1,\ x_2=1,\ x_3=1,\ \overline{x}_4=1)$

construct G:



$k=4$

Correctness: $\exists$ assignment that makes F true
$\iff$ $\exists$ indep set S for G of size $\geq k$.

Pf: $(\Longrightarrow)$ Let Δ be a sat. assignment for F.
Define a subset S as follows:

($\Rightarrow$)

Define a subset S as follows:

for each clause $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$,

pick some j s.t. α_{ij} is true.

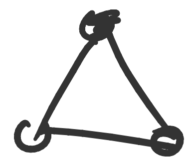
put v_{ij} in S.

Then $|S| = m = k$.

& S is an indep set

(check triangle edges ✓

cross edge ✓)



($\Leftarrow$) Let S be an indep set of G of size $\geq k$.

Define an assignment α as follows:

whenever v_{ij} is in S,

Set α_{ij} to true.

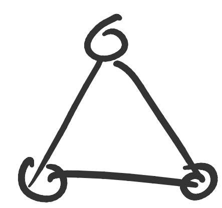
(Set all remaining vars arbitrarily.)

Then α is consistent:

if $\alpha_{ij} = \overline{\alpha_{i'j'}}$, can't have $v_{ij}, v_{i'j'}$ both in S because of cross edge

So can't set both $\alpha_{ij}, \alpha_{i'j'}$ true

α is satisfying:



for each triangle,

at most one v_{ij} is in S.

but since $|S| \geq k = m$,

exactly one v_{ij} is in S

so $\alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$ is true for α

$\forall i$. $\square$

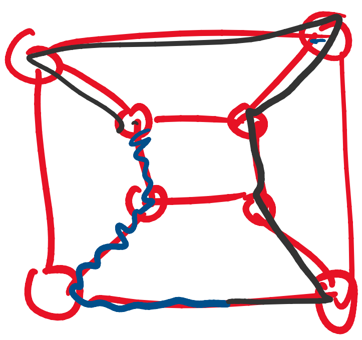
Cor Vertex-Cover is NP-complete.
Set-Cover " " .

Hamiltonian Cycle (HC)

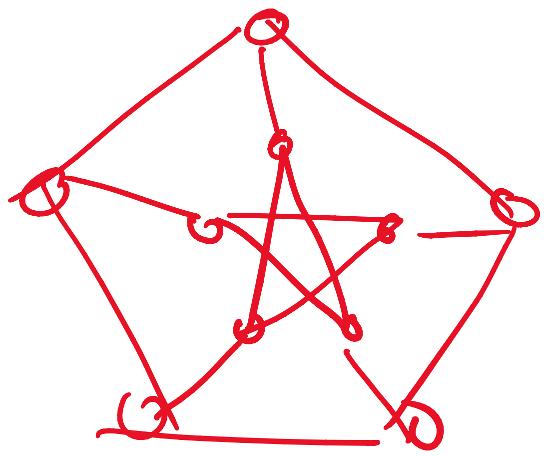
Input: graph $G=(V,E)$

Output: yes iff $\exists$ cycle visiting every vertex exactly once

e.g



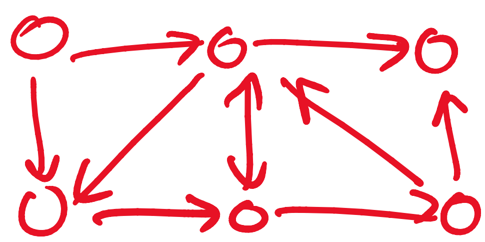
yes



no

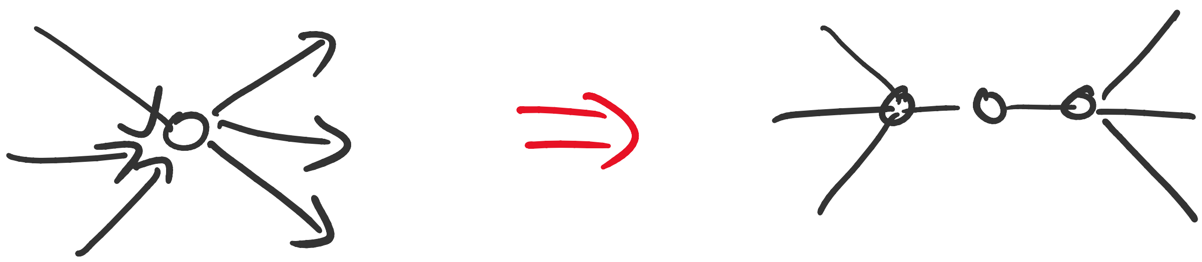
variants:

dir.



Ham path

Note: dir-HC $\leq_p$ undir-HC



Thm (Karp'72) HC is NP-complete.

Pf: ① $HC \in NP$ ✓

② Vertex-Cover $\le_p$ dir-HC:

Given input to Vertex Cover: undir graph $G=(V,E)$, integer k,

Construct input to dir-HC: dir graph G', as follows:

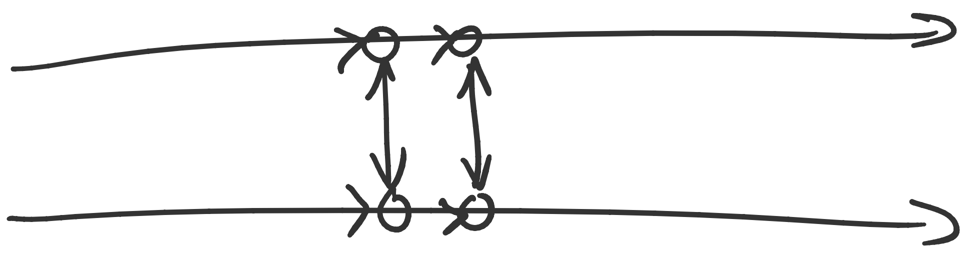
for each vertex $v \in V$,
draw a "line" ℓ_v

ℓ_v 

for each edge $uv \in E$,
add a gadget between ℓ_u & ℓ_v

ℓ_u

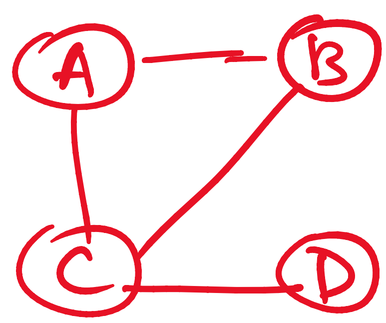
ℓ_v



(TO BE CONT'D)

⋮

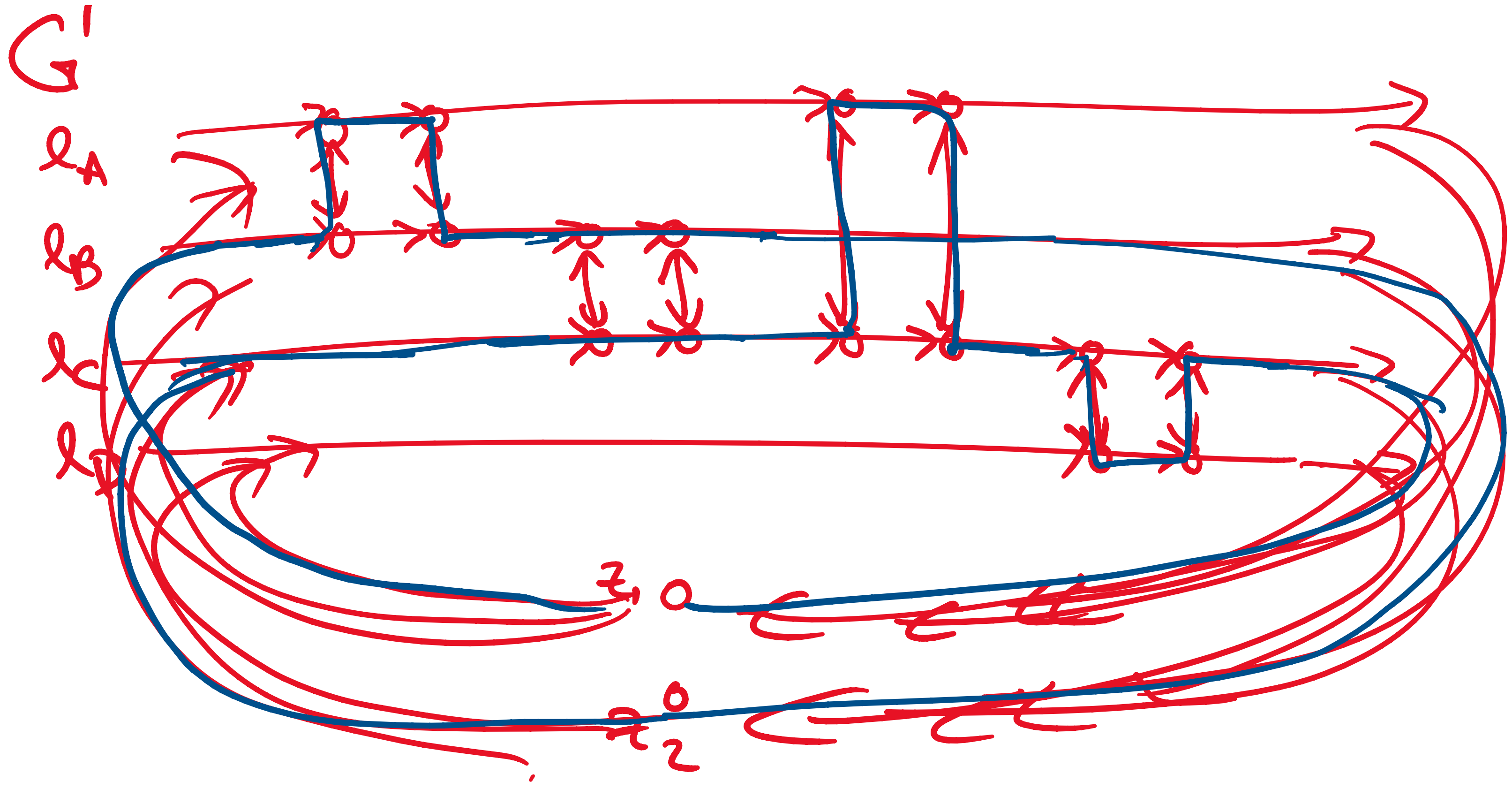
e.g. G



$k=2$

$\{B,C\}$

G'



G'
l_A
l_B
l_C
l_D
z_1
z_2